Disclaimer

The solution at hand was written in the course of the respective class at the University of Bonn. If not stated differently on top of the first page or the following website, the solution was prepared and handed in solely by me, Marvin Zanke. Anything in a different color than the ball pen blue is usually a correction that I or a tutor made. For more information and all my material, check:
https://www.physics-and-stuff.com/

I raise no claim to correctness and completeness of the given solutions! This equally applies to the corrections mentioned above.

This work by Marvin Zanke is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 4.0 International License.

16.10.2017 Advanced Quantum Field theory Exercise sheet 1 Flavio Lanke

P1) $\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - c\phi - \frac{m^2}{2}\phi^2 - \frac{g}{3!}\phi^3$

How to see that not bounded from below? ⇝ not bounded

a) 6 dimensions $\Rightarrow [\mathcal{L}] = M^6$, as $[S] = 0$, with

$$S = \int d^6x\, \mathcal{L}$$

$$[t] = [L] = M^{-1} \quad (\hbar = c = 1)$$

Small c doesn't change ϕ^3 behaviour; also multi-stable without cx?

⇝ Quantum fluctuations in pert. theory make system fall off!

but with cx metastable

$\rightsquigarrow [\phi] = M^2$ to achieve $[m^2\phi^2] = M^6$

$\rightsquigarrow [g\phi^3] = [g]M^6 \overset{!}{=} M^6 \Rightarrow [g] = 0$

In d dimensions ($d = 6 - 2\epsilon$ later)

$[\mathcal{L}] = M^d$

$\rightsquigarrow [\phi] = M^{\frac{d-2}{2}}$

$\rightsquigarrow [g\phi^3] = [g]\, M^{\frac{3d}{2}-3} \overset{!}{=} M^d \rightarrow [g] = M^{3-d/2}$

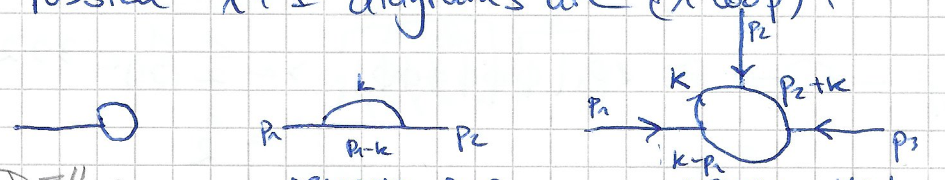
Choose $g \equiv \mu^{3-d/2} g$ (for simplicity no additional index)

$\rightsquigarrow$ in $d = 6 - 2\epsilon$ dimensions, need to choose

$x = \epsilon$

4 integrals ($\hat{=}$ diagrams)? ⇝ B_0, $p^2 B_1$ counted as two?

b) Possible 1PI diagrams are (1 loop):



$D = 4$ as $\propto \int dk \frac{k^5}{k^2}$ — renorm. prop. $\sim \int dk \frac{k^5}{k^4}$, $D = 2$ — renorm. vertex $\sim \int dk \frac{k^5}{k^6}$, $D = 0$

$-ig$ or ig for vertex? ⇝ if $-g$ int. term, then vertex is $-ig$

Feynman rules:
- vertex $= -ig$
- propagator $= \frac{i}{p^2 - m^2}$
- ext. particle $= 1$
- mom. conservation at vertices & sym. factors

Sup. degree of div.: general or for each diagram? → for each diagram ok!

Superficial degree of divergence:

d: dimension — V: Vertices

I: internal prop. — E: external lines

L: Loops

$$D = d \cdot L - 2I$$

$L = I - V + 1$ and $3V = 2I + E \Leftrightarrow I = \frac{3V - E}{2}$, $L = \frac{V - E + 2}{2}$

$\rightsquigarrow D = d + V(\frac{d}{2} - 3) + E(1 - \frac{d}{2}) \overset{d=6}{=} 6 - 2\epsilon$

$D>0$: divergent, where $D=0$: log. divergent

$D<0$: convergent, and thus all other diagrams convergent

$= -i\Sigma_1(p^2)$? $\rightsquigarrow$ just a notation

c)

Tadpole: $-\!\!\bigcirc \;\hat{=}\; \langle p_1 | \phi\phi\phi | 0\rangle \rightsquigarrow \frac{1}{3!}\cdot 3 = \frac{1}{2}$ Sym. factor

$$\rightsquigarrow -\!\!\bigcirc \underset{\epsilon\to 0}{=} \frac{-ig\mu^{3-d/2}}{2}\int\frac{d^dk}{(2\pi)^d}\frac{i}{k^2-m^2} = \frac{g\mu^{3-d/2}}{2}\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2-m^2}$$

Identify divergent terms by taylor exp.? Doesn't even matter which parts diverge? $\rightsquigarrow$ how many div. terms etc. (naiv

$$\underset{k^i=k^i_E}{\overset{k^0=ik^0_E}{=}} \frac{-ig\mu^{3-d/2}}{2}\int\frac{d^dk_E}{(2\pi)^d}\frac{1}{k_E^2+m^2} = \frac{-ig\mu^{3-d/2}}{2(2\pi)^d}\pi^{d/2}(m^2)^{d/2-1}\frac{\Gamma(1-d/2)}{\Gamma(1)}$$

MS Scheme? $\rightsquigarrow$ only $1/\epsilon$ coeff.

$$= \frac{-ig\mu^{3-d/2}}{2(4\pi)^{d/2}}(m^2)^{d/2-1}\,\Gamma(1-d/2) \overset{d=6-2\epsilon}{=} \frac{-ig\mu^{\epsilon}}{2(4\pi)^3}(4\pi)^{\epsilon}(m^2)^{2-\epsilon}\,\Gamma(\epsilon-2)$$

$$= \frac{-ig\mu^{-\epsilon}}{2(4\pi)^3}(m^2)^2\left(\frac{4\pi\mu^2}{m^2}\right)^{\epsilon}\left\{\frac{1}{2}\left(\frac{1}{\epsilon}+\Psi(3)+\mathcal{O}(\epsilon)\right)\right\}$$

Either calculate the 2-point fct. directly or will taylor exp.? $\rightsquigarrow$ could also taylor exp. first

$$= \frac{-ig\mu^{-\epsilon}}{4(4\pi)^3}m^4\left(\frac{4\pi\mu^2}{m^2}\right)^{\epsilon}\left(\frac{1}{\epsilon}+\frac{3}{2}-\gamma_E+\mathcal{O}(\epsilon)\right)$$

Still need to Wick-rotate or is the given formula for Minkowski metric? or $a^2=-m^2$ possible? $\rightsquigarrow$ because it's $+m^2$ in formula

$$c^{\epsilon} = e^{\epsilon\ln c} \approx 1+\epsilon\ln c+\mathcal{O}(\epsilon^2)$$

$$= \underbrace{\frac{-ig\mu^{-\epsilon}}{4(4\pi)^3}m^4}_{\text{Coefficient of } 1/\epsilon \text{ term}}\left(1+\epsilon\ln\left(\frac{4\pi\mu^2}{m^2}\right)+\mathcal{O}(\epsilon)\right)\left(\frac{1}{\epsilon}-\gamma_E+\frac{3}{2}+\mathcal{O}(\epsilon)\right)$$

Carry ϵ until end of calculation? $\rightsquigarrow$ until Wick rot.!

Propagator/self energy: $-\!\bigcirc\!- \;\hat{=}\; \langle p_1 | \phi_x\phi_x\phi_x\phi_y\phi_y\phi_y | p_2\rangle \sim \frac{1}{2}\frac{1}{(3!)^2}\cdot 3\cdot 3\cdot 2\cdot 2 = \frac{1}{2}$

Why use $\left(\frac{4\pi\mu^2}{m^2}\right)^{\epsilon} \simeq 1+\epsilon\ln(\ldots)$ in the end? $\rightsquigarrow$ to see the ϵ dep. better for $\epsilon \ll 1$

$$\rightsquigarrow -\!\bigcirc\!- = \frac{-g^2\mu^{6-d}}{2}\int\frac{d^dk}{(2\pi)^d}\frac{i}{(p-k)^2-m^2}\,\frac{i}{k^2-m^2}$$

$$= \frac{g^2\mu^{6-d}}{2}\int_0^1 dt\int\frac{d^dk}{(2\pi)^d}\frac{1}{\big[\underbrace{(1-t)(k^2-m^2)+t\{(p-k)^2-m^2\}}_{(*)}\big]^2}$$

$$(*) = k^2-m^2-\cancel{tk^2}+\cancel{tm^2}+tp^2+\cancel{tk^2}-2pkt-\cancel{tm^2}$$
$$= (k-pt)^2-p^2t^2+tp^2-m^2$$

$$\underset{=k}{\overset{k'=k-pt}{=}} \frac{g^2\mu^{6-d}}{2}\int_0^1 dt\int\frac{d^dk'}{(2\pi)^d}\frac{1}{[k^2-p^2t^2+tp^2-m^2]^2}$$

$$\overset{\text{Wick}}{=} \frac{ig^2\mu^{6-d}}{2}\int_0^1 dt\int\frac{d^dk_E}{(2\pi)^d}\frac{1}{[k_E^2+p^2t^2-p^2t+m^2]^2} \rightarrow a^2 = m^2-tp^2(1-t)$$

$$= \frac{ig^2\mu^{6-d}}{2(2\pi)^d}\int_0^1 dt \left\{ \pi^{d/2}(a^2)^{d/2-2}\,\frac{\Gamma(2-d/2)}{\Gamma(2)}\right.$$

$$= \frac{ig^2\mu^{6-d}}{2(4\pi)^{d/2}}\int_0^1 dt\,(a^2)^{d/2-2}\,\Gamma(2-d/2)$$

$$= \frac{ig^2\mu^{2\epsilon}}{2(4\pi)^3}(4\pi)^\epsilon\int_0^1 dt\,(a^2)^{1-\epsilon}\,\Gamma(\epsilon-1)$$

$$= \frac{ig^2\mu^{2\epsilon}}{2(4\pi)^3}(4\pi)^\epsilon\int_0^1 dt\,a^{2-2\epsilon}\left\{\frac{-1}{1}\left(\frac{1}{\epsilon}+\psi(2)+\mathcal{O}(\epsilon)\right)\right\}$$

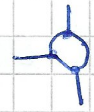
$$= \frac{-ig^2}{2(4\pi)^3}\int_0^1 dt\,a^2\left(\frac{4\pi\mu^2}{a^2}\right)^\epsilon\left(\frac{1}{\epsilon}+1-\gamma_E+\mathcal{O}(\epsilon)\right)$$

$$= \frac{-ig^2}{2(4\pi)^3}\int_0^1 dt\,a^2\left(1+\epsilon\ln\left(\frac{4\pi\mu^2}{a^2}\right)+\mathcal{O}(\epsilon^2)\right)\left(\frac{1}{\epsilon}+1-\gamma_E+\mathcal{O}(\epsilon)\right)$$

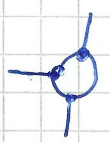
ϵ^0 term etc. Could still diverge? No UV div. though? Not easy to see. What diverges because of the dt integral? ⇝ max log. divergence in integral; in the end $\epsilon \to 0$

$$= \frac{-ig^2}{2(4\pi)^3}\int_0^1 dt\,a^2\left(\frac{1}{\epsilon}+\mathcal{O}(\epsilon^0)\right) = \frac{-ig^2}{2(4\pi)^3}\int_0^1 dt\,\frac{m^2-tp^2+t^2p^2}{\epsilon}+\mathcal{O}(\epsilon^0)$$

$$= \frac{-ig^2}{2(4\pi)^3}\frac{1}{\epsilon}\left\{m^2-\frac{1}{2}p^2+\frac{1}{3}p^2\right\}+\mathcal{O}(\epsilon^0) = \underbrace{\frac{-ig^2}{2(4\pi)^3}\left(m^2-\frac{p^2}{6}\right)}_{\text{Coefficient of } 1/\epsilon \text{ term}}\frac{1}{\epsilon}+\mathcal{O}(\epsilon^0)$$

Vertex:  $\hat{=} \langle p_1 | \phi_x\phi_x\phi_x\phi_y\phi_y\phi_y\phi_z\phi_z\phi_z | p_2 p_3\rangle \sim \frac{1}{3!}\frac{1}{(3!)^3}\,3\cdot3\cdot3\cdot3\cdot2\cdot2\cdot2\cdot2 = 1$

3 6 9 4 2 1

 $= \frac{ig^3\mu^{9-\frac{3}{2}d}}{1}\int\frac{d^dk}{(2\pi)^d}\frac{i}{(k^2-m^2)}\frac{i}{(p_2+k)^2-m^2}\frac{i}{(p_1-k)^2-m^2}$

How to see the integration domain in those integrals? ⇝ non zero only if sum $\delta(1-x-y-z)$ is 1 thus domain also needs to be controlled.

$$\overset{\text{3 Feynman parameters}}{=} \underset{\text{factor}}{2}\,\frac{g^3\mu^{9-\frac{3}{2}d}}{(2\pi)^d}\int_0^1 du\int_0^1 dv\int_0^1 dw\int d^dk\,\frac{\delta(u+v+w-1)}{\left[u\{(k^2-m^2)\}+v\{(p_2+k)^2-m^2\}+w\{(p_1-k)^2-m^2\}\right]^3}$$

$$= \frac{2g^3\mu^{9-\frac{3}{2}d}}{(2\pi)^d}\int_0^1 dv\int_0^{1-v} dw\int d^dk\,\frac{1}{\big[\underbrace{(1-v-w)(k^2-m^2)+v[(p_2+k)^2-m^2]+w[(p_1-k)^2-m^2]}_{(*)}\big]^3}$$

$$(*) = k^2-m^2\cancel{-vk^2}\cancel{+vm^2}\cancel{-wk^2}\cancel{+wm^2}+vp_2^2\cancel{+vk^2}+2p_2kv\cancel{-vm^2}$$
$$+wp_1^2\cancel{+wk^2}-2p_1kw\cancel{-wm^2}$$

$$= k^2-m^2+2k(p_2v-p_1w)+vp_2^2+wp_1^2$$

$$= \left(k+(p_2v-p_1w)\right)^2-(p_2v-p_1w)^2+vp_2^2+wp_1^2-m^2$$

$$k-(p_2v-p_1w) \quad = \frac{2g^3\mu^{9-3/2\,d}}{(2\pi)^d}\int_0^1 dv\int_0^{1-v} dw\int d^dk\,\frac{1}{[k^2-(p_2v-p_1w)^2+vp_2^2+wp_1^2-m^2]^3}$$

wick rot.

$$= \frac{-2ig^3\mu^{9-3/2\,d}}{(2\pi)^d}\int_0^1 dv\int_0^{1-v} dw\int d^dk_E\,\frac{1}{[k_E^2+\underbrace{(p_2v-p_1w)^2-vp_2^2-wp_1^2+m^2}_{\equiv b^2}]^3}$$

$$= \frac{-2ig^3\mu^{9-3/2\,d}}{(2\pi)^d}\int_0^1 dv\int_0^{1-v} dw\left\{\pi^{d/2}\,(b^2)^{d/2-3}\,\frac{\Gamma(3-d/2)}{\underbrace{\Gamma(3)}_{2}}\right\}$$

$$= \frac{-ig^3\mu^{9-3/2\,d}}{(4\pi)^{d/2}}\int_0^1 dv\int_0^{1-v} dw\left\{(b^2)^{d/2-3}\,\Gamma(3-d/2)\right\}$$

$$= \frac{-ig^3\mu^{3\epsilon}}{(4\pi)^3}(4\pi)^{\epsilon}\int_0^1 dv\int_0^{1-v} dw\; b^{-2\epsilon}\,\Gamma(\epsilon)$$

$$= \frac{-ig^3\mu^{\epsilon}}{(4\pi)^3}\int_0^1 dv\int_0^{1-v} dw\left(\frac{4\pi\mu^2}{b^2}\right)^{\epsilon}\left\{\frac{1}{\epsilon}-\gamma_E+\mathcal{O}(\epsilon)\right\}$$

$$= \frac{-ig^3\mu^{\epsilon}}{(4\pi)^3}\int_0^1 dv\int_0^{1-v} dw\left(1+\epsilon\ln\left(\frac{4\pi\mu^2}{b^2}\right)+\mathcal{O}(\epsilon^2)\right)\left(\frac{1}{\epsilon}-\gamma_E+\mathcal{O}(\epsilon)\right)$$

$$= \frac{-ig^3\mu^{\epsilon}}{(4\pi)^3}\int_0^1 dv\int_0^{1-v} dw\left(\frac{1}{\epsilon}+\mathcal{O}(\epsilon^0)\right) = \frac{-ig^3\mu^{\epsilon}}{(4\pi)^3}\int_0^1 dv\,(1-v)\left(\frac{1}{\epsilon}+\mathcal{O}(\epsilon^0)\right)$$

$$= \frac{-ig^3\mu^{\epsilon}}{(4\pi)^3}\left(1-\frac{1}{2}\right)\left(\frac{1}{\epsilon}+\mathcal{O}(\epsilon^0)\right) = \underbrace{\frac{-ig^3\mu^{\epsilon}}{2(4\pi)^3}}_{\text{coefficient of } 1/\epsilon \text{ term}}\left(\frac{1}{\epsilon}+\mathcal{O}(\epsilon^0)\right)$$

Still ϵ in μ in front of eq.?

d) $\mathcal{L} = \frac{1}{2}(\partial\phi_r)^2 - c\phi_r - \frac{m_r^2}{2}\phi_r^2 - \frac{g_r}{3!}\phi_r^3$

$+ \frac{1}{2}\delta_Z(\partial\phi_r)^2 - \frac{1}{2}\delta_m\phi_r^2 - \delta_c\phi_r - \frac{\delta_g}{3!}\phi_r^3$

Why can we just add term? $+-0$ actually?
→ It's just $\pm$ and Z, m_0 free of choice, even though we later fix δ_m etc.

with $\delta_Z = Z_\phi - 1$, $\delta_m = m^2 Z_\phi - m^2$

$\delta_c = \sqrt{Z_\phi}\, c - c_r$, $\delta_g = Z_\phi^{3/2} g - g_r$

$\phi = \sqrt{Z_\phi}\,\phi_r$

If we calculate diagrams with the new $\mathcal{L}$, how do they now cancel the divergences?

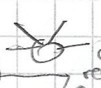
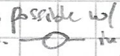
$-i\delta_c = \frac{-ig\mu^{-\epsilon}}{4(4\pi)^3} m^4 =$ 

$i(p^2\delta_Z - \delta_m) = \left(\frac{-ig^2}{2(4\pi)^3}\left(m^2 - \frac{p^2}{6}\right)\right)\cdot(-1)$

Super renormalizable?
→ renorm → many (inf.) diagrams but renorm. in finite constants (similar diagrams absorbed in similar fashion)

$\Rightarrow ip^2\delta_Z = \frac{-ig^2}{12(4\pi)^3} p^2 =$ 

$-i\delta_m = \frac{ig^2}{2(4\pi)^3} m^2 =$ 

Do we need renorm. only for 1PI diagrams or do we also calculate a regular loop diagram?
→ e.g. loop in 1st order; e.g.   also possible w/ renorm.  ϕ^3 2 loops yet different, not covered by 2× the 1 loop correction

$ip^2\delta_Z$ for  2
→ from derivative p^2

Why regularize/renormalize if in the end the result is the same?
→ also gives new finite contribution

Why self energy / prop. renorm. if we know exact prop. by $\sum_n x^n$ geom. series?
→ in this $\frac{1}{p^2 - m^2 - i\Sigma(p^2)}$ there's in $\Sigma(p^2)$ still the pert. series, which we know would have to work!