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18.12.2017 Advanced Quantum Field theory Exercise 9

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P13) Let the β-function be given by

$$\beta(g) = g(a^2 - g^2) \quad \text{w/ } a \text{ being a constant}$$

valid for every theory where g instead of λ?

a) Using the initial condition $g(t=0) = g_0$, we sketch $\beta(g)$ vs. g:

$\bar{g}(t=0)$?! NOT $g(t=0)$ → doesn't matter, just solving p.d.e and make a difference to integration var.

Stable fixpoint (little deviation → +a or -a)

We also have

$$\frac{\partial}{\partial t} \bar{g}(t) = \beta(\bar{g})$$

For the behaviour of $\bar{g}(t)$ for $t \to \infty$, we denote:

- Starting w/ $\bar{g}(t=0) = g_0 \in (a, \infty)$, we have a negative $\beta(g)$, thus we have a decreasing $\bar{g}(t)$ (see P.D.E.)
  → decreases until $\beta(g) = 0$, then we have a fixpoint
  → looking at the graph, we have the fixpoint at $\beta(a) = 0$
  $\Leftrightarrow \bar{g} = a$

- Starting with $\bar{g}(t=0) = g_0 \in (0, a)$, we have a positive $\beta(g)$, thus a growing $\bar{g}(t)$
  → increases until $\beta(g) = 0$
  → fixpoint at $\bar{g} = a$

- For $\bar{g}(t=0) = g_0 = 0$, we have a fixpoint at $\bar{g} = 0$, as $\beta(0) = 0$

- Analogue for $a_0 < 0$, yielding fixpoints in $\bar{g} = -a$

b) $\beta(\bar{g}) = \bar{g}(a^2 - \bar{g}^2) = \frac{\partial}{\partial t}\bar{g}(t)$

$$\Leftrightarrow 1 = \frac{1}{\beta(\bar{g})}\frac{\partial \bar{g}(t)}{\partial t} \Leftrightarrow \int_0^t dt' = \int_0^t dt' \frac{1}{\beta(\bar{g}(t'))}\frac{d\bar{g}(t')}{dt'}$$

$$\Leftrightarrow t = \int_{\bar{g}(0)}^{\bar{g}(t)} d\bar{g}' \frac{1}{\beta(\bar{g}')}$$

$$\frac{1}{x(c^2-x^2)} = \frac{A}{x} + \frac{B}{c^2-x^2} \rightsquigarrow A(c^2-x^2)+Bx = 1 = Ac^2 - Ax^2 + Bx$$

$$\Rightarrow A = \frac{1}{c^2}, \quad B = \frac{x}{c^2}$$

$$\Leftrightarrow t = \int_{\bar{g}(0)=g_0}^{\bar{g}(t)} d\bar{g}' \left\{ \frac{1}{a^2}\frac{1}{\bar{g}'} + \frac{1}{a^2}\frac{\bar{g}'}{a^2-\bar{g}'^2} \right\}$$

$$= \left[\frac{1}{a^2}\log(\bar{g}') - \frac{1}{2a^2}\log(a^2-\bar{g}'^2) \right] \Bigg|_{g_0}^{\bar{g}(t)}$$

$g_0 \neq 0$

$$= \frac{1}{a^2}\left\{ \log(\bar{g}(t)) - \frac{1}{2}\log(a^2-\bar{g}(t)^2) - \log(g_0) + \frac{1}{2}\log(a^2-g_0^2) \right\}$$

$$\rightsquigarrow e^{a^2 t} = e^{\log(\bar{g}(t))}\left(e^{\log(a^2-\bar{g}^2(t))}\right)^{-1/2}\left(e^{\log(g_0)}\right)^{-1}\left(e^{\log(a^2-g_0^2)}\right)^{1/2}$$

$$= \bar{g}(t)\,(a^2-\bar{g}^2(t))^{-1/2}\, g_0^{-1}\,(a^2-g_0^2)^{1/2}$$

Where case $\bar{g}(t) \to 0$ excluded? $\rightsquigarrow$ dividing by $g(a^2-g^2)$ $\rightsquigarrow$ always 0 and not defined.

$$\Leftrightarrow \frac{g_0 e^{a^2 t}}{\sqrt{a^2-g_0^2}} = \frac{\bar{g}(t)}{\sqrt{a^2-\bar{g}^2(t)}}$$

$$\Leftrightarrow \frac{g_0^2 e^{2a^2 t}}{a^2-g_0^2} = \frac{\bar{g}^2(t)}{a^2-\bar{g}^2(t)}$$

$$\Leftrightarrow \bar{g}^2(t)\left\{1 + \frac{g_0^2 e^{2a^2 t}}{a^2-g_0^2}\right\} = a^2\frac{g_0^2 e^{2a^2 t}}{a^2-g_0^2}$$

Why logs $= \log \mathrm{P}/\mu$?

$$\Leftrightarrow \bar{g}^2(t) = a^2 \frac{\frac{g_0^2 e^{2a^2 t}}{a^2-g_0^2}}{1+\frac{g_0^2 e^{2a^2 t}}{a^2-g_0^2}} = a^2\frac{1}{\frac{a^2-g_0^2}{g_0^2 e^{2a^2 t}} + 1}$$

$$\rightsquigarrow \bar{g}(t) \xrightarrow{t\to\infty} \pm a$$

P14)

a) $\beta(g) = -b(g-a)$ with $b>0$ (a simple zero)

$g(t=0) = g_0$

Again $\frac{\partial}{\partial t}\bar{g} = -b(\bar{g}-a) = \beta(\bar{g})$

$g(t) \xrightarrow{t\to\infty} a$ instead of $\bar{g}(t)$?

$$\Longleftrightarrow \int_0^t \frac{1}{\beta(\bar{g}(t'))}\frac{\partial \bar{g}}{\partial t}dt' = \int_0^t 1\, dt'$$

Name for the diff eq ?

$$\Longleftrightarrow t = \int_{\bar{g}(0)}^{\bar{g}(t)} d\bar{g}' \frac{1}{(-b(\bar{g}-a))} = -\frac{1}{b}\left\{\log(\bar{g}-a)\right\}\Big|_{g_0}^{\bar{g}(t)}$$

$$= -\frac{1}{b}\left\{\log(\bar{g}(t)-a) - \log(g_0-a)\right\}$$

$b>0 \Rightarrow$ UV-stable fixpoint or where does UV-stable f.p. go in? $\rightarrow$ UV ($t\to\infty$) and UV-stable means that it's the same from both sides ($g_0<a$, $g_0>a$).

$$\Longleftrightarrow e^{-bt} = e^{\log(\bar{g}(t)-a)}\left(e^{\log(g_0-a)}\right)^{-1}$$

$$= \frac{(\bar{g}(t)-a)}{g_0-a}$$

$$\Longleftrightarrow (g_0-a)e^{-bt} = \bar{g}(t)-a$$

$$\Longleftrightarrow \bar{g}(t) = a + (g_0-a)e^{-bt} \xrightarrow{t\to\infty} a \text{ for } b>0$$

exponentially

b) Now consider double or higher zeroes

$\beta(g) = -b(g-a)^n$, $b>0$, $n>1$

$$\rightsquigarrow t = \int_{\bar{g}(0)}^{\bar{g}(t)} d\bar{g}' \frac{1}{(-b(\bar{g}-a)^n)} = -\frac{1}{b}\left\{-\frac{1}{n-1}(\bar{g}-a)^{-(n-1)}\right\}\Big|_{g_0}^{\bar{g}(t)}$$

$$= \frac{1}{b(n-1)}\left\{(\bar{g}(t)-a)^{-n+1} - (g_0-a)^{-n+1}\right\}$$

$$\Longleftrightarrow (n-1)bt = (\bar{g}(t)-a)^{-n+1} - (g_0-a)^{-n+1}$$

$$\Longleftrightarrow (\bar{g}(t)-a)^{-n+1} = (g_0-a)^{-n+1} + (n-1)bt$$

$$\Longleftrightarrow \frac{1}{(\bar{g}(t)-a)^{n-1}} = \frac{1}{(g_0-a)^{n-1}} + (n-1)bt$$

$$= \frac{1+(g_0-a)^{n-1}(n-1)bt}{(g_0-a)^{n-1}}$$

$$\Leftrightarrow (\bar{g}(t)-a)^{n-1} = \frac{(g_0-a)^{n-1}}{1+(g_0-a)^{n-1}(n-1)bt}$$

$$\Leftrightarrow \bar{g}(t)-a = \frac{g_0-a}{\left(1+(g_0-a)^{n-1}(n-1)bt\right)^{1/n-1}}$$

$$\Leftrightarrow \bar{g}(t) = a + \frac{g_0-a}{\left(1+(g_0-a)^{n-1}(n-1)bt\right)^{\frac{1}{n-1}}} \xrightarrow{t\to\infty} a$$

with $\sim \left(\frac{1}{t}\right)^{\frac{1}{n-1}}$

P15) $$\mathcal{L}_{QCD} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a + \sum_f \bar{q}_f \left(i\gamma^\mu \left(\partial_\mu - igA^b_\mu \frac{\lambda_b}{2} \right) - m_f \right) q_f$$

w/ $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc} A^b_\mu A^c_\nu$

q and $\bar{q}$ count as one fermion? → yes, summed over $\bar{u}$ ... u

"current" quark mass m_f? → no sense here, no counterterms → bare mass

q_f Dirac / Color spinor → both?

(a) As we will be taking derivatives of the form $\frac{\delta^3}{\delta\bar{q}\,\delta q\,\delta A}$, the responsible term for the vertex is

$$\mathcal{L}_g = \sum_f \bar{q}_f \left(i\gamma^\mu \left(-igA^b_\mu \frac{\lambda_b}{2} \right) \right) q_f$$
$$= \sum_e \bar{q}_e \gamma^\nu g A^b_\nu \frac{\lambda_b}{2} q_e = g \sum_e (\bar{q}_e(x))_{\beta' k'} (\gamma^\nu)_{\beta'\beta} A^b_\nu(x) \left(\frac{\lambda_b}{2}\right)_{k'k} (q_e(x))_{\beta k}$$

The corresponding action integral is thus given by

$$S = \int d^4x \left\{ g \sum_e \bar{q}_e(x) \gamma^\nu A^b_\nu(x) \frac{\lambda_b}{2} q_e(x) \right\}$$

why are there 3 gluon field vector potentials A^a_μ physically?

mom. space / $(2\pi)^{-4}$ factors cancelled by outer contractions

$$= \int d^4x\, g \sum_e \int d^4q_1\, \bar{q}_e(q_1)_{\beta' k'} e^{-iq_1 x} (\gamma^\nu)_{\beta'\beta}$$
$$\times \int d^4q_2\, A^b_\nu(q_2) e^{-iq_2 x} \left(\frac{\lambda_b}{2}\right)_{k'k}$$
$$\times \int d^4q_3\, q_e(q_3)_{\beta k} e^{-iq_3 x}$$

Is this the path integral formalism? only take the specific parts of the Lagrangian? not really a generating functional?

$$\rightsquigarrow \frac{\delta^3 S}{\delta(\bar{q}_{f'}(p'))_{\alpha' j'}\, \delta(q_f(p))_{\alpha j}\, \delta A^a_\mu(q)}$$

$$= \frac{\delta^2}{\delta(\bar{q}_{f'}(p'))_{\alpha' j'}\, \delta(q_f(p))_{\alpha j}} \Big\} \int d^4x\, g \sum_e \int d^4q_1\, \bar{q}_e(q_1)_{\beta' k'} e^{-iq_1 x} (\gamma^\mu)_{\beta'\beta}$$
$$\times e^{-iqx} \left(\frac{\lambda_a}{2}\right)_{k'k}$$
$$\times \int d^4q_3\, q_e(q_3)_{\beta k} e^{-iq_3 x} \Big\}$$

where exactly did the factors $(2\pi)^{-4}$ go?

$$= \frac{\delta}{\delta(\bar{q}_{f'}(p'))_{\alpha' j'}} \Big\} \int d^4x\, g \int d^4q_1\, \bar{q}_f(q_1)_{\beta' k'} e^{-iq_1 x} (\gamma^\mu)_{\beta'\alpha}$$
$$\times e^{-iqx} \left(\frac{\lambda_a}{2}\right)_{k' j}$$
$$\times e^{-ipx} \Big\}$$

where to write the 2nd and 3rd index of q, $\bar{q}$?

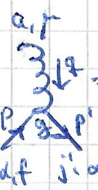
$\bar{q}$ outg. particle w/ reversed mom.

$$= \int d^4x \, g \, \delta_{f'f} \, e^{ip'x} \, (\gamma^\mu)_{\alpha'\alpha} \, e^{-iqx} \left(\frac{\lambda_a}{2}\right)_{j'j} e^{-ipx}$$

$$= g \, \delta_{f'f} \, (\gamma^\mu)_{\alpha'\alpha} \left(\frac{\lambda_a}{2}\right)_{j'j} \int d^4x \, e^{-ix(p+q-p')}$$

$$= g \left(\frac{\lambda_a}{2}\right)_{j'j} (\gamma^\mu)_{\alpha'\alpha} \, \delta_{f'f} \, (2\pi)^4 \, \delta^{(4)}(p+q-p')$$

$\rightsquigarrow$ multiplying with (i) yields



$$= ig \left(\frac{\lambda_a}{2}\right)_{j'j} (\gamma^\mu)_{\alpha'\alpha} \, \delta_{f'f} \, (2\pi)^4 \, \delta^{(4)}(p+q-p')$$

How to do the formal way w/o mult. with (i) in the end?

b) the first part of $\mathcal{L}_{QCD}$ is the only possible term that can be responsible for the 3 gluon vertex

$$\mathcal{L}^{(1)}_{QCD} = -\tfrac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a = -\tfrac{1}{4} (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu)$$
$$\times (\partial^\mu A^\nu_a - \partial^\nu A^\mu_a + g f_{ade} A^\mu_d A^\nu_e)$$

the terms we are looking for are $\sim A^3$, thus

inconsistency between summing over spatial indices? ✓

→ trivial metric, can lower or raise indices as we want

$$\mathcal{L}_{3g} = -\tfrac{1}{4} \{ (\partial_\mu A^a_\nu) g f_{ade} A^\mu_d A^\nu_e - (\partial_\nu A^a_\mu) g f_{ade} A^\mu_d A^\nu_e$$
$$+ g f^{abc} A^b_\mu A^c_\nu (\partial^\mu A^\nu_a) - g f^{abc} A^b_\mu A^c_\nu (\partial^\nu A^\mu_a) \}$$

$$= -\tfrac{g}{2} \{ (\partial_\mu A^a_\nu) f_{abc} A^\mu_b A^\nu_c - (\partial_\nu A^a_\mu) f_{abc} A^\mu_b A^\nu_c \}$$

$$\left(= -\tfrac{g}{2} \{ \partial_\mu A^a_\nu - \partial_\nu A^a_\mu \} f_{abc} A^\mu_b A^\nu_c \right)$$

$$= -\tfrac{g}{2} \{ (\partial_\nu A^a_\mu) f_{abc} A^\nu_b A^\mu_c - (\partial_\nu A^a_\mu) f_{abc} A^\mu_b A^\nu_c \}$$

if A are fields (bosons), do they always commute w/ everything?

$$= -\tfrac{g}{2} \{ (\partial_\nu A^a_\mu) f_{acb} A^\nu_c A^\mu_b - (\partial_\nu A^a_\mu) f_{abc} A^\mu_b A^\nu_c \}$$

$$= g (\partial_\nu A^a_\mu) f_{abc} A^\mu_b A^\nu_c = g f_{abc} g^{\mu\delta} (\partial^\nu A^a_\mu) A^b_\delta A^c_\nu$$

$$= g f_{def} g^{\epsilon\delta} (\partial^\kappa A^d_\epsilon) A^e_\delta A^f_\kappa$$

We will now express the arguments of the fields we are deriving with respect to and derive the Lagrangian instead of the action. It was obvious from the exercise a), that the only difference will be the factor $(2\pi)^4 \times \delta^{(4)}$ ("mom. cons.")

$$i \frac{\delta^3 \mathcal{L}_{3g}}{\delta(A^a_\mu)\delta(A^b_\nu)\delta(A^c_\rho)} = i \frac{\delta^2}{\delta(A^b_\nu)\delta(A^c_\rho)} \{ g f_{aef} g^{\mu\delta} (-ip^\kappa) A^e_\delta A^f_\kappa$$
$$+ g f_{daf} g^{\epsilon\mu} (\partial^\kappa A^d_\epsilon) A^f_\kappa$$
$$+ g f_{dea} g^{\epsilon\delta} (\partial^\mu A^d_\epsilon) A^e_\delta \}$$

$$= ig \frac{\delta}{\delta(A^c_\rho)} \Big\{ f_{abf}\, g^{\mu\nu} (-ip^\kappa) A^f_\kappa + f_{aeb}\, g^{\mu\sigma} (-ip^\nu) A^e_\sigma$$
$$+ f_{baf}\, g^{\nu\mu} (-iq^\kappa) A^f_\kappa + f_{dab}\, g^{\epsilon\mu} (\partial^\nu A^d_\epsilon)$$
$$+ f_{bea}\, g^{\nu\sigma} (-iq^\mu) A^e_\sigma + f_{dba}\, g^{\epsilon\nu} (\partial^\mu A^d_\epsilon) \Big\}$$

$$= ig \Big\{ f_{abc}\, g^{\mu\nu} (-ip^\rho) + f_{acb}\, g^{\mu\rho} (-ip^\nu) + f_{bac}\, g^{\nu\mu} (-iq^\rho) + f_{cab}\, g^{\rho\mu} (-ir^\nu)$$
$$+ f_{bca}\, g^{\nu\rho} (-iq^\mu) + f_{cba}\, g^{\rho\nu} (-ir^\mu) \Big\}$$

$$= g \Big\{ f_{abc}\, \underline{g^{\mu\nu}}\, p^\rho - f_{abc}\, g^{\mu\rho} p^\nu - f_{abc}\, \underline{g^{\nu\mu}}\, q^\rho + f_{abc}\, g^{\rho\mu} r^\nu$$
$$+ f_{abc}\, g^{\nu\rho} q^\mu - f_{abc}\, g^{\rho\nu} r^\mu \Big\}$$

$$= g f_{abc} \Big\{ g^{\mu\nu} (p-q)^\rho + g^{\mu\rho} (r-p)^\nu + g^{\nu\rho} (q-r)^\mu \Big\}$$

multiply w/ $(2\pi)^4 \delta^{(4)}$

c) We take a look at $\mathcal{L}_{QCD}^{(1)}$ again

Terms $\sim A^2$ in propagator?

$$\mathcal{L}_{QCD}^{(1)} = -\frac{1}{4}\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c\right) \times \left(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a + g f_{ade} A_d^\mu A_e^\nu\right)$$

And take into account terms $\sim A^4$

$$\rightsquigarrow \mathcal{L}_{4g} = -\frac{1}{4}\left\{ g f^{abc} A_\mu^b A_\nu^c\, g f_{ade} A_d^\mu A_e^\nu \right\}$$

$$= -\frac{g^2}{4} f^{abc} f_{ade} A_\mu^b A_\nu^c A_d^\mu A_e^\nu$$

$$= -\frac{g^2}{4} f_{abc} f_{ade} A_\mu^b A_\nu^c A_\sigma^d A_\tau^e g^{\sigma\mu} g^{\tau\nu}$$

$$= -\frac{g^2}{4} f_{abc} f_{ade} g^{\mu\sigma} g^{\nu\tau} A_\mu^b A_\sigma^d A_\nu^c A_\tau^e$$

$$= -\frac{g^2}{4} f_{fgh} f_{fjk} g^{\alpha\gamma} g^{\beta\delta} A_\alpha^g A_\gamma^j A_\beta^h A_\delta^k$$

✓

Faster way? Combine terms earlier? $\rightsquigarrow$ not done but definitely 24 terms

$$i\frac{\delta^4 \mathcal{L}_{4g}}{\delta(A_\mu^a)\delta(A_\nu^b)\delta(A_\rho^c)\delta(A_\sigma^d)}$$

$$= -\frac{ig^2}{4}\frac{\delta^3}{\delta(A_\mu^a)\delta(A_\nu^b)\delta(A_\rho^c)}\Big\{ f_{fdh} f_{fjk} g^{\sigma\gamma} g^{\beta\delta} A_\gamma^j A_\beta^h A_\delta^k + f_{fgh} f_{fdk} g^{\alpha\sigma} g^{\beta\delta} A_\alpha^g A_\beta^h A_\delta^k$$

$$+ f_{fgd} f_{fjk} g^{\alpha\gamma} g^{\sigma\delta} A_\alpha^g A_\gamma^j A_\delta^k + f_{fgh} f_{fjd} g^{\alpha\gamma} g^{\beta\sigma} A_\alpha^g A_\gamma^j A_\beta^h$$

$$= -\frac{ig^2}{4}\frac{\delta^2}{\delta(A_\mu^a)\delta(A_\nu^b)}\Big\{ f_{fdh} f_{fck} g^{\sigma\rho} g^{\beta\delta} A_\beta^h A_\delta^k + f_{fdc} f_{fjk} g^{\sigma\gamma} g^{\rho\delta} A_\gamma^j A_\delta^k$$

$$+ f_{fdh} f_{fjc} g^{\sigma\gamma} g^{\beta\rho} A_\gamma^j A_\beta^h$$

$$+ f_{fch} f_{fdk} g^{\rho\sigma} g^{\beta\delta} A_\beta^h A_\delta^k + f_{fgc} f_{fdk} g^{\alpha\sigma} g^{\rho\delta} A_\alpha^g A_\delta^k$$

$$+ f_{fgh} f_{fdc} g^{\alpha\sigma} g^{\beta\rho} A_\alpha^g A_\beta^h$$

$$+ f_{fcd} f_{fjk} g^{\rho\gamma} g^{\sigma\delta} A_\gamma^j A_\delta^k + f_{fgd} f_{fck} g^{\alpha\rho} g^{\sigma\delta} A_\alpha^g A_\delta^k$$

$$+ f_{fgd} f_{fjc} g^{\alpha\gamma} g^{\sigma\rho} A_\alpha^g A_\gamma^j$$

$$+ f_{fch} f_{fjd} g^{\rho\gamma} g^{\beta\sigma} A_\gamma^j A_\beta^h + f_{fgh} f_{fcd} g^{\alpha\rho} g^{\beta\sigma} A_\alpha^g A_\beta^h$$

$$+ f_{fgc} f_{fjd} g^{\alpha\gamma} g^{\rho\sigma} A_\alpha^g A_\gamma^j \Big\}$$

$$
\begin{aligned}
= \frac{-ig^2}{4}\frac{\delta}{\delta(A^a_\mu)}\Big\{ & f_{fdb}f_{fck}\, g^{\sigma\rho} g^{\nu\delta} A^k_\delta + f_{fdh} f_{fcb}\, g^{\sigma\rho} g^{\beta\nu} A^h_\beta + f_{fdc} f_{fbk}\, g^{\sigma\nu} g^{\rho\delta} A^k_\delta \\
& + f_{fdc} f_{fjb}\, g^{\rho\sigma} g^{\gamma\nu} A^j_\gamma + f_{fdh} f_{fbc}\, g^{\sigma\nu} g^{\beta\rho} A^h_\beta + f_{fdb} f_{fjc}\, g^{\sigma\rho} g^{\nu\gamma} A^j_\gamma \\
& + f_{fcb} f_{fdk}\, g^{\rho\sigma} g^{\nu\delta} A^k_\delta + f_{fch} f_{fdb}\, g^{\rho\sigma} g^{\beta\nu} A^h_\beta + f_{fbc} f_{fdk}\, g^{\nu\sigma} g^{\rho\delta} A^k_\delta \\
& + f_{fgc} f_{fdb}\, g^{\alpha\sigma} g^{\rho\nu} A^g_\alpha + f_{fbh} f_{fdc}\, g^{\nu\sigma} g^{\beta\rho} A^h_\beta + f_{fgb} f_{fdc}\, g^{\alpha\sigma} g^{\nu\rho} A^g_\alpha \\
& + f_{fcd} f_{fbk}\, g^{\rho\nu} g^{\sigma\delta} A^k_\delta + f_{fcd} f_{fjb}\, g^{\rho\gamma} g^{\sigma\nu} A^j_\gamma + f_{fbd} f_{fck}\, g^{\nu\rho} g^{\sigma\delta} A^k_\delta \\
& + f_{fgd} f_{fcb}\, g^{\alpha\rho} g^{\sigma\nu} A^g_\alpha + f_{fbd} f_{fjc}\, g^{\nu\gamma} g^{\sigma\rho} A^j_\gamma + f_{fgd} f_{fbc}\, g^{\alpha\nu} g^{\sigma\rho} A^g_\alpha \\
& + f_{fch} f_{fbd}\, g^{\rho\nu} g^{\beta\sigma} A^h_\beta + f_{fcb} f_{fjd}\, g^{\rho\gamma} g^{\nu\sigma} A^j_\gamma + f_{fbh} f_{fcd}\, g^{\nu\rho} g^{\beta\sigma} A^h_\beta \\
& + f_{fgb} f_{fcd}\, g^{\alpha\rho} g^{\nu\sigma} A^g_\alpha + f_{fbc} f_{fjd}\, g^{\nu\gamma} g^{\rho\sigma} A^j_\gamma + f_{fgc} f_{fbd}\, g^{\alpha\nu} g^{\rho\sigma} A^g_\alpha \Big\}
\end{aligned}
$$

$$
\begin{aligned}
= \frac{-ig^2}{4}\Big\{ & 2g^{\mu\nu}g^{\rho\sigma} f_{fac}f_{fbd} + 2g^{\mu\nu}g^{\rho\sigma} f_{fad}f_{fbc} + 2g^{\mu\nu}g^{\rho\sigma} f_{fad}f_{fbc} \\
& + 2g^{\mu\nu}g^{\rho\sigma} f_{fac}f_{fbd} + 2g^{\mu\rho}g^{\nu\sigma} f_{fab}f_{fcd} - 2g^{\mu\rho}g^{\nu\sigma} f_{fad}f_{fbc} \\
& + 2g^{\mu\rho}g^{\nu\sigma} f_{fab}f_{fcd} - 2g^{\mu\rho}g^{\nu\sigma} f_{fad}f_{fbc} - 2g^{\mu\sigma}g^{\nu\rho} f_{fab}f_{fcd} \\
& - 2g^{\mu\sigma}g^{\nu\rho} f_{fac}f_{fbd} - 2g^{\mu\sigma}g^{\nu\rho} f_{fab}f_{fcd} - 2g^{\mu\sigma}g^{\nu\rho} f_{fac}f_{fbd} \Big\}
\end{aligned}
$$

$$
\begin{aligned}
= -ig^2 \Big\{ & g^{\mu\nu}g^{\rho\sigma} f_{fad}f_{fbc} + g^{\mu\nu}g^{\rho\sigma} f_{fac}f_{fbd} + g^{\mu\rho}g^{\nu\sigma} f_{fab}f_{fcd} \\
& - g^{\mu\rho}g^{\nu\sigma} f_{fad}f_{fbc} - g^{\mu\sigma}g^{\nu\rho} f_{fab}f_{fcd} - g^{\mu\sigma}g^{\nu\rho} f_{fac}f_{fbd} \Big\}
\end{aligned}
$$

$$
\begin{aligned}
= -ig^2 \Big\{ & f_{fac}f_{fbd}\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}\right) + f_{fad}f_{fbc}\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma}\right) \\
& + f_{fab}f_{fcd}\left(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}\right) \Big\}
\end{aligned}
$$

multiply w/ $(2\pi)^4 \delta^{(4)}$