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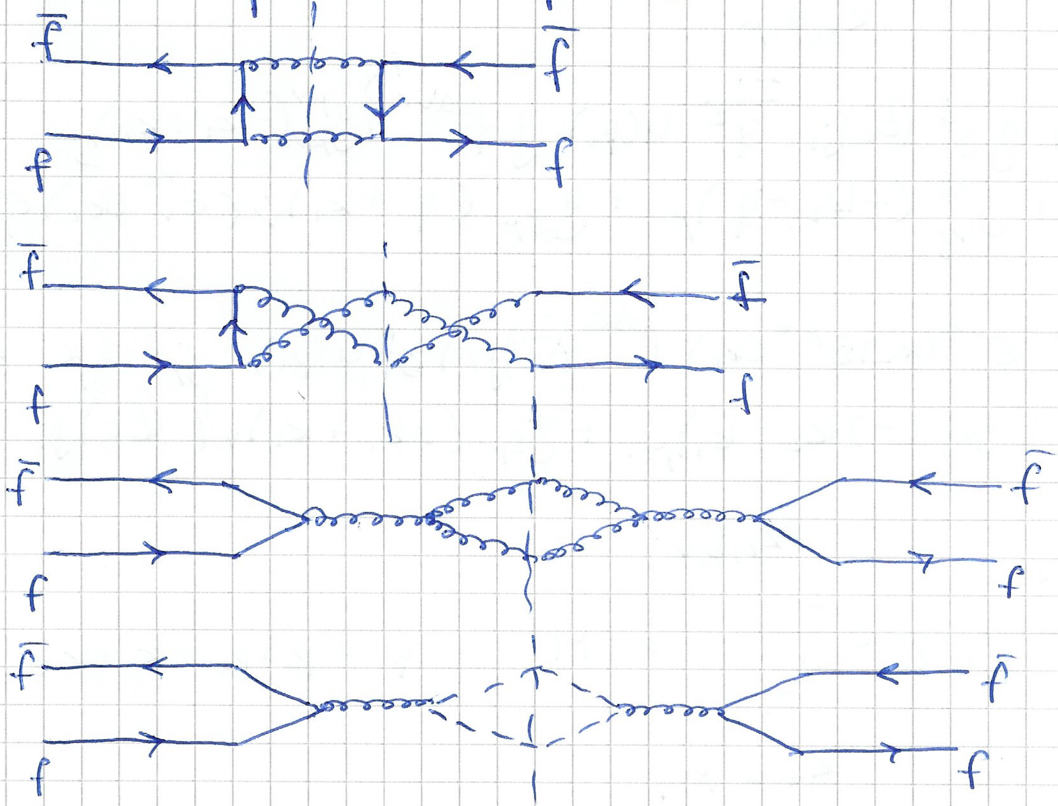
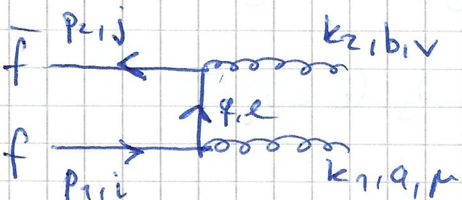
08.01.2018 Advanced Quantum Field Theory Exercise 11 Marina Zanker

P1(6) a) the possible 1-loop diagrams are



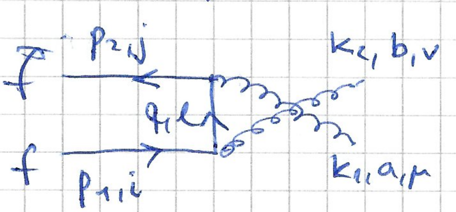
b)



$= iM_1$, $q_1 = p_1 - k_1$

$$= \bar v(p_2)\, ig\gamma_\nu (t_b)_{je} \frac{i(\not q_1 + m)}{q_1^2 - m^2} ig\gamma_\mu (t_a)_{ei}\, u(p_1)\, \epsilon^{\nu *}(k_2)\, \epsilon^{\mu *}(k_1)$$

$$= \frac{-ig^2}{q_1^2 - m^2} \epsilon^{\nu *}(k_2)\, \epsilon^{\mu *}(k_1)\, \bar v(p_2)\gamma_\nu(\not q_1 + m)\gamma_\mu u(p_1)\, (t_b t_a)_{ji}$$



$= iM_2$, $p_1 = q_2 + k_2$, $q_2 + p_2 = k_1$

$$= \bar v(p_2)\, ig\gamma_\mu (t_a)_{je} \frac{i(\not q_2 + m)}{q_2^2 - m^2} ig\gamma_\nu (t_b)_{ei}\, u(p_1)\, \epsilon^{\nu *}(k_2)\, \epsilon^{\mu *}(k_1)$$

$$= \frac{-ig^2}{q_2^2 - m^2} \epsilon^{\nu *}(k_2)\, \epsilon^{\mu *}(k_1)\, \bar v(p_2)\gamma_\mu(\not q_2 + m)\gamma_\nu u(p_1)\, (t_a t_b)_{ji}$$

Where from eq. (2)?
→ insert the prop. into (1). Cut the diagram and replace by those on-shell prop.

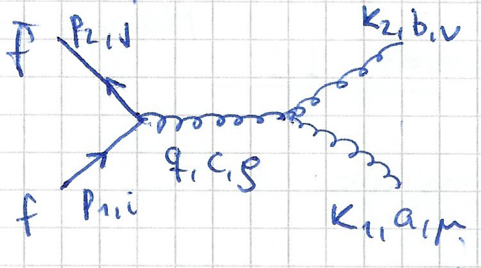
1-loop diagr. for l.h.s., but not on shell → no ghosts as should be on shell if cut?
→ can go on shell but not physical

Also u-channel in the r.h.s. of the cut? Same diagr. as t-channel then?
→ interference terms between those

Ghost int. med. state not of the form on ex. w/ A as int. med.?
→ doesn't matter, int. med. state can be "everything"

Where in (2) is $\epsilon_\mu \epsilon_\nu$? $T^{ab}_{\mu\nu}$ should have no free μ, ν indices?
→ contracted w/ $P^{\mu\nu}$ and in those are the $\epsilon^\mu \epsilon^\nu$

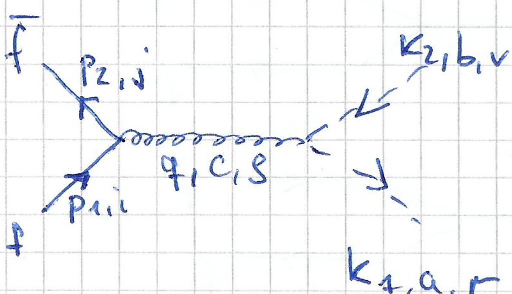
Color index on spinors? Or do they just commute?



$= iM_3 \quad , \quad q = k_1 + k_2 = p_1 + p_2$

$$= \bar v(p_2)\, i g \gamma^\rho (t_c)_{ji} \frac{-i}{q^2} u(p_1)\, g f^{abc} \{ g_{\mu\nu}(-k_1+k_2)_\rho + g_{\nu\rho}(-k_2-q)_\mu + g_{\rho\mu}(q+k_1)_\nu \} \epsilon^{\nu *}(k_2)\epsilon^{\mu *}(k_1)$$

$$= \frac{g^2}{q^2} \epsilon^{\nu *}(k_2)\epsilon^{\mu *}(k_1) f^{abc}\, \bar v(p_2)\gamma^\rho u(p_1) \{ g_{\mu\nu}(k_2-k_1)_\rho + g_{\nu\rho}(-2k_2-k_1)_\mu + g_{\rho\mu}(2k_1+k_2)_\nu \} (t_c)_{ji}$$



$= iM_4 , \quad q = p_1 + p_2 = k_1 + k_2$

$$= \bar v(p_2)\, i g \gamma^\rho (t_c)_{ji} \frac{-i}{q^2} u(p_1) \left(-g f^{acb} (k_1)_\rho\right)$$

$$= \frac{g^2}{q^2} \bar v(p_2)\gamma^\rho u(p_1) f^{abc} (k_1)_\rho (t_c)_{ji}$$

Choose k_1 or k_2 for ghost vertex?
→ in Peskin outgoing w/ (-) sign

never ✓ 3-gauge vertex in abelian theory? we only knows SU(2)→U(1) → QED

c) As there is no 3-gauge-vertex in the abelian
i) case, we only need the first 2 diagrams.

$$k_1^\mu T^{ab}_{\mu\nu} = k_1^\mu \Big\{ \frac{-g^2}{q_1^2-m^2} \bar{v}(p_2)\gamma_\nu (\not{q}_1+m)\gamma_\mu u(p_1)(t_b t_a)_{ji}$$
$$- \frac{g^2}{q_2^2-m^2} \bar{v}(p_2)\gamma_\mu (\not{q}_2+m)\gamma_\nu u(p_1)(t_a t_b)_{ji} \Big\}$$

$$= -g^2 \bar{v}(p_2) \Big\{ \frac{1}{q_1^2-m^2} \gamma_\nu (\not{q}_1+m) \not{k}_1 (t_b t_a)_{ji}$$
$$+ \frac{1}{q_2^2-m^2} \not{k}_1 (\not{q}_2+m)\gamma_\nu (t_a t_b)_{ji} \Big\} u(p_1)$$

$$\overset{\substack{q_1 = p_1 - k_1 \\ q_2 = k_1 - p_2}}{=} -g^2 \bar{v}(p_2) \Big\{ \frac{1}{q_1^2-m^2} \gamma_\nu (\not{q}_1+m)(\not{p}_1-\not{q}_1)(t_b t_a)_{ji}$$
$$+ \frac{1}{q_2^2-m^2} (\not{q}_2+\not{p}_2)(\not{q}_2+m)\gamma_\nu (t_a t_b)_{ji} \Big\} u(p_1)$$

$$= -g^2 \bar{v}(p_2) \Big\{ \frac{1}{q_1^2-m^2} \gamma_\nu (m^2-q_1^2)(t_b t_a)_{ji}$$
$$+ \frac{1}{q_2^2-m^2} (q_2^2-m^2)\gamma_\nu (t_a t_b)_{ji} \Big\} u(p_1)$$

$$= g^2 \bar{v}(p_2)\gamma_\nu \Big\{ (t_b t_a)_{ji} - (t_a t_b)_{ji} \Big\} u(p_1)$$

$$= g^2 \bar{v}(p_2)\gamma_\nu \underbrace{[t_b, t_a]_{ji}}_{= \begin{cases} i f^{bac}(t_c)_{ji} \\ 0 \text{ for abelian} \end{cases}} u(p_1)$$

This vanishes in the abelian case

$$\rightsquigarrow \text{RHS} = \frac{1}{2}\int dP_2\, T^{ab}_{\mu\nu} T^{ab\,*}_{\mu'\nu'} \left(-g^{\mu\mu'} + \frac{k_1^\mu \eta^{\mu'} + k_1^{\mu'}\eta^\mu}{k_1\cdot\eta}\right)$$
$$\times \left(-g^{\nu\nu'} + \frac{k_2^\nu \eta^{\nu'} + k_2^{\nu'}\eta^\nu}{k_2\cdot\eta}\right)$$

can we use $k_2^\nu T^{ab}_{\mu\nu} = 0$ as well? ✓ remember, commutes

$$= \frac{1}{2}\int dP_2\, T^{ab}_{\mu\nu} T^{ab\,*}_{\mu'\nu'} g^{\mu\mu'} g^{\nu\nu'}$$

No ghosts $(S^{ab} S^{ab\,*})$ needed to fulfill LHS. ✓

ii) Additionally:

$$k_1^\mu \Big\{ \frac{-ig^2}{q^2} f^{abc}\, \bar{v}(p_2)\gamma^\rho u(p_1) \big\{ g_{\mu\nu}(k_2-k_1)_\rho - g_{\nu\rho}(2k_2+k_1)_\mu + g_{\rho\mu}(2k_1+k_2)_\nu \big\} (t_c)_{ji}$$

$$= \frac{-ig^2}{q^2} f^{abc}\, \bar{v}(p_2)\gamma^\rho u(p_1) \big\{ (k_1)_\nu (k_2-k_1)_\rho - g_{\nu\rho}(2k_2\cdot k_1 + k_1^2) + (k_1)_\rho (2k_1+k_2)_\nu \big\} (t_c)_{ji}$$

$(k_1+k_2)^2 = (p_1+p_2)^2 = 2k_1k_2 = q^2$

$$= \frac{-ig^2}{q^2} f^{abc}\, \bar{v}(p_2) \big\{ (k_1)_\nu (\not{k}_2 - \not{k}_1) - \gamma_\nu q^2 + \not{k}_1 (2k_1+k_2)_\nu \big\} u(p_1)(t_c)_{ji}$$

We also have

$$-S^{ab}(k_2)_\nu = \frac{ig^2}{q^2} \bar{v}(p_2)\gamma^\rho u(p_1) f^{abc} (k_1)_\rho (t_c)_{ji} (k_2)_\nu$$

$$= \frac{ig^2}{q^2} f^{abc}\, \bar{v}(p_2) \not{k}_1 u(p_1) (t_c)_{ji} (k_2)_\nu$$

and in total

$$k_1^\mu T^{ab}_{\mu\nu} = ig^2 f^{bac}\, \bar{v}(p_2)\gamma_\nu (t_c)_{ji} u(p_1)$$

$$- \frac{ig^2}{q^2} f^{abc}\, \bar{v}(p_2) \big\{ (k_1)_\nu (\not{k}_2 - \not{k}_1) - \gamma_\nu q^2 + \not{k}_1 (2k_1+k_2)_\nu \big\} u(p_1)(t_c)_{ji}$$

$$= -ig^2 f^{abc}\, \bar{v}(p_2)\gamma_\nu u(p_1)(t_c)_{ji}$$

$$+ ig^2 f^{abc}\, \bar{v}(p_2)\gamma_\nu u(p_1)(t_c)_{ji}$$

$$- \frac{ig^2}{q^2} f^{abc}\, \bar{v}(p_2) \big\{ (k_1)_\nu (\not{k}_2 - \not{k}_1) + \not{k}_1 (2k_1+k_2)_\nu \big\} u(p_1)(t_c)_{ji}$$

$$= -\frac{ig^2}{q^2} f^{abc}\, \bar{v}(p_2) \big\{ (k_1)_\nu \not{k}_1 + (k_1)_\nu \not{k}_2 + (k_2)_\nu \not{k}_1 \big\} u(p_1)(t_c)$$

$$= \frac{-ig^2}{q^2} f^{abc}\, \bar{v}(p_2) \big\{ (k_1)_\nu \overbrace{(\not{k}_1 + \not{k}_2)}^{= \not{p}_1 + \not{p}_2 \to (m-m)} + (k_2)_\nu \not{k}_1 \big\} u(p_1)(t_c)_{ji}$$

$$= \frac{-ig^2}{q^2} f^{abc}\, \bar{v}(p_2) \not{k}_1 u(p_1)(t_c)_{ji}(k_2)_\nu$$

Closed loop
→ factor $(-1)^2$
⇝ (−) sign between
$TT^\dagger = SS^\dagger$

d) $\frac{1}{2}\int d\rho_2\, T^{ab}_{\mu\nu} T^{ab*}_{\mu'\nu'} P^{\mu\mu'}(k_1) P^{\nu\nu'}(k_2)$

$= \frac{1}{2}\int d\rho_2\, T^{ab}_{\mu\nu} T^{ab*}_{\mu'\nu'} \left\{-g^{\mu\mu'} + \frac{k_1^\mu \eta_1^{\mu'} + k_1^{\mu'}\eta_1^\mu}{k_1\cdot\eta_1}\right\} \times \left\{-g^{\nu\nu'} + \frac{k_2^\nu \eta_2^{\nu'} + k_2^{\nu'}\eta_2^\nu}{k_2\cdot\eta_2}\right\}$

Different η_μ's? ✓
→ yes

$= \frac{1}{2}\int d\rho_2 \Big\{ T^{ab}_{\mu\nu} T^{ab*}_{\mu'\nu'} g^{\mu\mu'} g^{\nu\nu'} - g^{\mu\mu'} T^{ab}_{\mu\nu} T^{ab*}_{\mu'\nu'} \frac{k_2^\nu\eta_2^{\nu'} + k_2^{\nu'}\eta_2^\nu}{k_2\cdot\eta_2}$

$- g^{\nu\nu'} T^{ab}_{\mu\nu} T^{ab*}_{\mu'\nu'} \frac{k_1^\mu\eta_1^{\mu'} + k_1^{\mu'}\eta_1^\mu}{k_1\cdot\eta_1}$

$+ T^{ab}_{\mu\nu} T^{ab*}_{\mu'\nu'} \frac{k_1^\mu\eta_1^{\mu'} + k_1^{\mu'}\eta_1^\mu}{k_1\cdot\eta_1} \frac{k_2^\nu\eta_2^{\nu'} + k_2^{\nu'}\eta_2^\nu}{k_2\cdot\eta_2} \Big\}$

Now, make use of $k_1^\mu T^{ab}_{\mu\nu} = -S^{ab} k_{2\nu}$

$T^{ab}_{\mu\nu} k_2^\nu = -S^{ab} k_{1\mu}$

k real? ✓
then just c.c. for the other eq.?
k real, 4 vector

and therefore as well

$k_1^\mu T^{ab}_{\mu\nu} k_2^\nu = 0 = k_1^{\mu'} T^{ab}_{\mu'\nu'} k_2^{\nu'}$

and $-g^{\mu\mu'} T^{ab}_{\mu\nu} T^{ab*}_{\mu'\nu'} k_2^\nu \eta_2^{\nu'} = -g^{\mu\mu'}(-S^{ab} k_{1\mu}) T^{ab*}_{\mu'\nu'}\eta_2^{\nu'}$

$= S^{ab}(-S^{ab(*)} k_{2\nu'})\eta_2^{\nu'}$

$= -\eta_2\cdot k_2\, S^{ab}(S^{ab})^*$

$-g^{\mu\mu'} T^{ab}_{\mu\nu} T^{ab*}_{\mu'\nu'} k_2^{\nu'}\eta_2^\nu = -g^{\mu\mu'}(-S^{ab*} k_{1\mu'}) T^{ab}_{\mu\nu}\eta_2^\nu$

$= S^{ab*}(-S^{ab} k_{2\nu})\eta_2^\nu$

$= -\eta_2\cdot k_2\, S^{ab}(S^{ab})^*$

$-g^{\nu\nu'} T^{ab}_{\mu\nu} T^{ab*}_{\mu'\nu'} k_1^\mu \eta_1^{\mu'} = -\eta_1\cdot k_1\, S^{ab}(S^{ab})^*$

$-g^{\nu\nu'} T^{ab}_{\mu\nu} T^{ab*}_{\mu'\nu'} k_1^{\mu'}\eta_1^\mu = -\eta_1\cdot k_1\, S^{ab}(S^{ab})^*$

$T^{ab}_{\mu\nu} T^{ab*}_{\mu'\nu'} k_1^\mu \eta_1^{\mu'} k_2^{\nu'}\eta_2^\nu = S^{ab}(S^{ab})^* k_{2\nu} k_{1\mu'} \eta_1^{\mu'}\eta_2^\nu$

$= S^{ab}(S^{ab})^*(\eta_1\cdot k_1)(\eta_2\cdot k_2)$

$T^{ab}_{\mu\nu} T^{ab*}_{\mu'\nu'} k_1^{\mu'}\eta_1^\mu k_2^\nu \eta_2^{\nu'} = S^{ab}(S^{ab})^*(\eta_1\cdot k_1)(\eta_2\cdot k_2)$

$$= \frac{1}{2}\int d\Omega_2 \left\{ T^{ab}_{\mu\nu} T^{ab*}_{\mu'\nu'} g^{\mu\mu'} g^{\nu\nu'} - 2(S^{ab})(S^{ab})^* - 2(S^{ab})(S^{ab})^* + 2(S^{ab})(S^{ab})^* \right\}$$

$$= \int d\Omega_2 \left\{ \frac{1}{2} T^{ab}_{\mu\nu} T^{ab*}_{\mu'\nu'} g^{\mu\mu'} g^{\nu\nu'} - (S^{ab})(S^{ab})^* \right\}$$