Disclaimer

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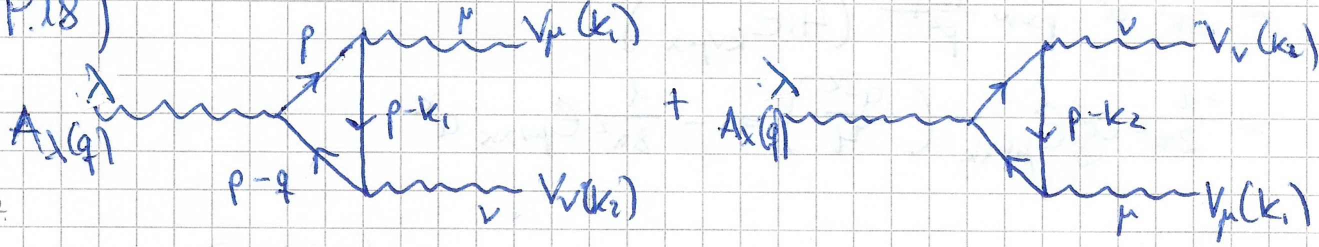
I raise no claim to correctness and completeness of the given solutions! This equally applies to the corrections mentioned above.

20.01.2018 Advanced Quantum Field Theory Exercise 12 Marvin Zeinke

P.18)



Let $T_{\mu\nu\lambda}(a)$ be the sum of the upper diagrams with shifted momentum $p \to p + a^{(\prime)}$, $a' = \alpha k_2 + (\alpha-\beta) k_1$

$a = \alpha k_1 + (\alpha - \beta) k_2$, i.e.

$$T_{\mu\nu\lambda}(a) = T^{(1)}_{\mu\nu\lambda}(a) + T^{(2)}_{\mu\nu\lambda}(a)$$

$$T^{(1)}_{\mu\nu\lambda}(a) = -\int \frac{d^4p}{(2\pi)^4} \operatorname{tr}\left\{ \gamma_\mu \frac{1}{\not p + \not a - m} \gamma_\lambda \gamma_5 \frac{1}{\not p + \not a - \not q - m} \gamma_\nu \frac{1}{\not p + \not a - \not k_1 - m} \right\}$$

$$T^{(2)}_{\mu\nu\lambda}(a) = -\int \frac{d^4p}{(2\pi)^4} \operatorname{tr}\left\{ \gamma_\nu \frac{1}{\not p + \not a - m} \gamma_\lambda \gamma_5 \frac{1}{\not p + \not a - \not q - m} \gamma_\mu \frac{1}{\not p + \not a - \not k_2 - m} \right\}$$

We instantly notice that $T^{(2)}_{\mu\nu\lambda}(a) = T^{(1)}_{\mu\nu\lambda}(a) \begin{pmatrix} \mu \leftrightarrow \nu \\ k_1 \leftrightarrow k_2 \\ a \leftrightarrow a' \end{pmatrix}$

Using $\int d^4p \{ t(p+a) - t(p) \} = 2i\pi^2 a^\rho \lim_{p\to\infty} p^2 p_\rho t(p)$, we find

$$\underbrace{T_{\mu\nu\lambda}(a) - T_{\mu\nu\lambda}(0)}_{=\Delta_{\mu\nu\lambda}(a)} = T^{(1)}_{\mu\nu\lambda}(a) - T^{(1)}_{\mu\nu\lambda}(0) + T^{(2)}_{\mu\nu\lambda}(a) - T^{(2)}_{\mu\nu\lambda}(0)$$

and

$$T^{(1)}_{\mu\nu\lambda}(a) - T^{(1)}_{\mu\nu\lambda}(0) = -\frac{1}{(2\pi)^4} \cdot 2i\pi^2 a^\rho \lim_{p\to\infty} p^2 p_\rho \left\{ \operatorname{tr}\left(\gamma_\mu \frac{1}{\not p - m} \gamma_\lambda \gamma_5 \times \frac{1}{\not p - \not q - m} \gamma_\nu \frac{1}{\not p - \not k_1 - m} \right) \right\}$$

$$\overset{\text{for lim } p^2 \gg m^2, q^2, k_i^2}{=} -\frac{i}{8\pi^2} a^\rho \lim_{p\to\infty} p^2 p_\rho \frac{\operatorname{tr}(\gamma_\mu \not p \gamma_\lambda \gamma_5 \not p \gamma_\nu \not p)}{p^6}$$

$$\operatorname{tr}(\gamma_\mu \not p \gamma_\lambda \gamma_5 \not p \gamma_\nu \not p) = \operatorname{tr}(\gamma_\lambda \gamma_5 \not p \gamma_\nu \not p \gamma_\mu \not p)$$

$$= \operatorname{tr}(\gamma_\lambda \gamma_5 \not p \gamma_\nu \not p (2 g_{\mu\kappa} - \gamma_\kappa \gamma_\mu) p^\kappa)$$

$$= \underline{2 p_\mu \operatorname{tr}(\gamma_\lambda \gamma_5 \not p \gamma_\nu \not p)} - p^2 \operatorname{tr}(\gamma_\lambda \gamma_5 \not p \gamma_\nu \gamma_\mu)$$

sym in 2 indices from $\not p$, antisym from ϵ-tensor ⇒ vanishes

what does it mean for the photon? to be an axial current? Why axial coupling in QED? ⇝ not the photon, general axial coupling particle

why change in $T(a) - T(0)$ and then still have to calculate $\hat\Delta_{\mu\nu\lambda}$?

Also just pol vector for axial photon? ⇝ yes, except for γ_5 same Feynman rules

Proof that we need a trace for closed fermion loops?

where from Feynman rule for axial current?

factors of i and e for vertices and prop.? ⇝ can introduce later

why not use the same a for both integrals? ⇝ not what we want to show here?

in lecture kept k_1 in trace? ⇝ check power counting first, keep const, lin div., etc and then calculate further/simplify

$$= -\frac{i}{8\pi^2} a^\rho \lim_{p\to\infty} \frac{p_\rho}{p^4}\left(-p^2 \operatorname{tr}(\gamma_\lambda\gamma_5 \not{p}\, \gamma_\nu\gamma_\mu)\right)$$

$$= \frac{i}{8\pi^2} a^\rho \lim_{p\to\infty} \frac{p_\rho p^\kappa}{p^2}\left(4i\epsilon_{\kappa\nu\mu\lambda}\right)$$

$$= \frac{-1}{2\pi^2} a_\rho \epsilon_{\mu\nu\lambda\kappa}\left(\frac{g^{\rho\kappa}}{4}\right) = -\frac{1}{8\pi^2}\epsilon_{\mu\nu\lambda\kappa} a^\kappa$$

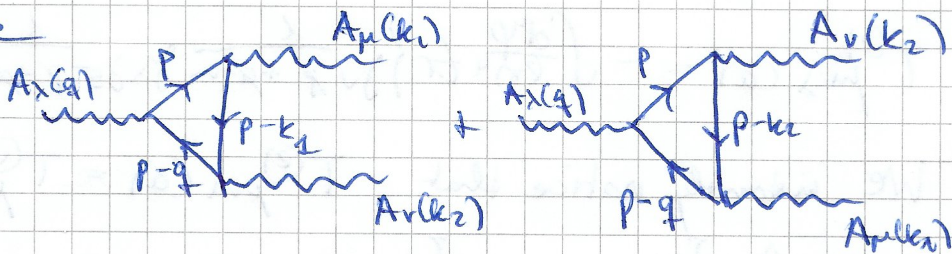
$$\rightsquigarrow T_{\mu\nu\lambda}(a) - T_{\mu\nu\lambda}(0) = T^{(1)}_{\mu\nu\lambda}(a) - T^{(1)}_{\mu\nu\lambda}(0) + \begin{pmatrix} \mu \leftrightarrow \nu \\ k_1 \leftrightarrow k_2 \\ a \leftrightarrow a' \end{pmatrix}$$

$$= -\frac{1}{8\pi^2}\epsilon_{\mu\nu\lambda\kappa} a^\kappa - \frac{1}{8\pi^2}\epsilon_{\nu\mu\lambda\kappa} a'^\kappa$$

$$= -\frac{1}{8\pi^2}\epsilon_{\mu\nu\lambda\kappa}\left\{\left(\alpha k_1^\kappa + (\alpha-\beta)k_2^\kappa\right) - \left(\alpha k_2^\kappa + (\alpha-\beta)k_1^\kappa\right)\right\}$$

$$= -\frac{1}{8\pi^2}\epsilon_{\mu\nu\lambda\kappa}\left\{-\beta k_2^\kappa + \beta k_1^\kappa\right\} = -\frac{\beta}{8\pi^2}\epsilon_{\mu\nu\lambda\kappa}(k_1^\kappa - k_2^\kappa)$$

b) We now have



Using the same prescription

$$T'_{\mu\nu\lambda}(a) - T'_{\mu\nu\lambda}(0) = T^{(1)'}_{\mu\nu\lambda}(a) - T^{(1)'}_{\mu\nu\lambda}(0) + T^{(2)'}_{\mu\nu\lambda}(a) - T^{(2)'}_{\mu\nu\lambda}(0)$$

We find

$$T^{(1)'}_{\mu\nu\lambda}(a) = -\int\frac{d^4p}{(2\pi)^4}\operatorname{tr}\left\{\gamma_\mu\gamma_5 \frac{1}{\not{p}+\not{a}-m}\gamma_\lambda\gamma_5\frac{1}{\not{p}+\not{a}-\not{q}-m}\gamma_\nu\gamma_5\frac{1}{\not{p}+\not{a}-\not{k}_1-m}\right\}$$

$$\rightsquigarrow T^{(1)'}_{\mu\nu\lambda}(a) - T^{(1)'}_{\mu\nu\lambda}(0) = -\frac{1}{(2\pi)^4}\, 2i\pi^2 a^\rho \lim_{p\to\infty} p^2 p_\rho \frac{\operatorname{tr}(\gamma_\mu\gamma_5\not{p}\,\gamma_\lambda\gamma_5\not{p}\,\gamma_\nu\gamma_5\not{p})}{p^6}$$

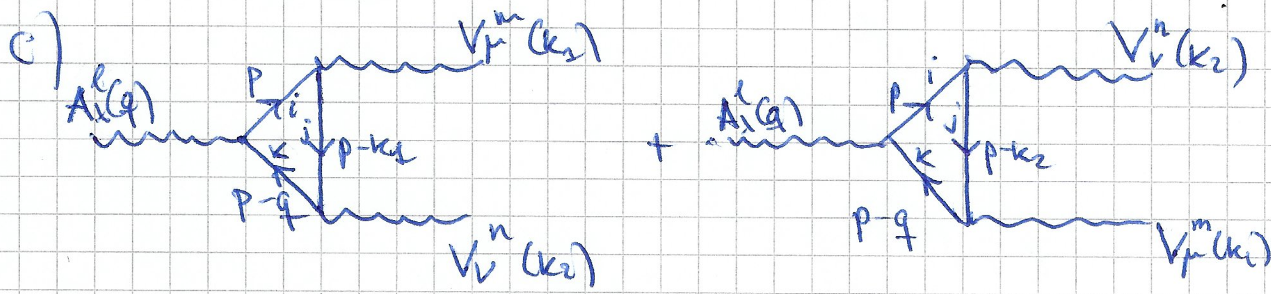
$$\Big|\ \operatorname{tr}(\gamma_\mu\gamma_5\not{p}\,\gamma_\lambda\gamma_5\not{p}\,\gamma_\nu\gamma_5\not{p})$$

$$= -\operatorname{tr}(\gamma_\mu\not{p}\,\gamma_\lambda(\gamma_5)^3\not{p}\,\gamma_\nu\not{p})$$

$$= \operatorname{tr}(\gamma_\mu\not{p}\,\gamma_\lambda\gamma_5\not{p}\,\gamma_\nu\not{p}) \triangleq \text{same trace}$$

$$\rightsquigarrow T'_{\mu\nu\lambda}(a) - T'_{\mu\nu\lambda}(0) = -\frac{\beta}{8\pi^2}\epsilon_{\mu\nu\lambda\kappa}(k_1^\kappa - k_2^\kappa) \quad \text{same result !}$$

c)



what for the spinors on the current?
$\rightsquigarrow$ contraction of incoming / outgoing field?

$$\underbrace{\tilde{T}_{\mu\nu\lambda}(a) - \tilde{T}_{\mu\nu\lambda}(0)}_{=\tilde{\Delta}_{\mu\nu\lambda}(a)} = \tilde{T}^{(1)}_{\mu\nu\lambda}(a) - \tilde{T}^{(1)}_{\mu\nu\lambda}(0) + T^{(2)}_{\mu\nu\lambda}(a) - T^{(2)}_{\mu\nu\lambda}(0)$$

What's an isovector current?

with the same prescription and $\tilde{T}^{(2)}_{\mu\nu\lambda}(a) = T^{(1)}_{\mu\nu\lambda}(a)\begin{pmatrix}\mu\leftrightarrow\nu\\ k_1\leftrightarrow k_2\\ a\leftrightarrow a'\\ n\leftrightarrow m\end{pmatrix}$

We find

Start at any point of the diagram? $\rightsquigarrow$ yes

$$\tilde{T}^{(1)}_{\mu\nu\lambda}(a) = -\int\frac{d^4p}{(2\pi)^4}\,tr\Big\{\gamma_\mu (t^m)_{ji}\frac{1}{\not p+\not a-m}\gamma_\lambda\gamma_5 (t^l)_{ik}\frac{1}{\not p+\not a-\not q-m}$$
$$\times\gamma_\nu (t^n)_{kj}\frac{1}{\not p+\not a-\not k_1-m}\Big\}$$
$$= -tr(t^m t^l t^n)\int\frac{d^4p}{(2\pi)^4}\Big\{\gamma_\mu\frac{1}{\not p+\not a-m}\gamma_\lambda\gamma_5\frac{1}{\not p+\not a-\not q-m}$$
$$\times\gamma_\nu\frac{1}{\not p+\not a-\not k_1-m}\Big\}$$
$$= tr(t^m t^l t^n)\, T^{(1)}_{\mu\nu\lambda}(a)$$

$$\rightsquigarrow \tilde{T}^{(1)}_{\mu\nu\lambda}(a) - T^{(1)}_{\mu\nu\lambda}(0) = tr(t^m t^l t^n)\Big\{T^{(1)}_{\mu\nu\lambda}(a) - T^{(1)}_{\mu\nu\lambda}(0)\Big\}$$
$$= tr(t^m t^l t^n)\Big\{-\frac{1}{8\pi^2}\epsilon_{\mu\nu\lambda\kappa}a^\kappa\Big\}$$

$$\rightsquigarrow \tilde{T}_{\mu\nu\lambda}(a) - \tilde{T}_{\mu\nu\lambda}(0) = tr(t^m t^l t^n)\Big\{-\frac{1}{8\pi^2}\epsilon_{\mu\nu\lambda\kappa}a^\kappa\Big\}$$
$$+ tr(t^n t^l t^m)\Big\{-\frac{1}{8\pi^2}\epsilon_{\nu\mu\lambda\kappa}a'^\kappa\Big\}$$
$$= -\frac{1}{8\pi^2}\epsilon_{\mu\nu\lambda\kappa}\Big\{tr(t^m t^l t^n)a^\kappa - tr(t^n t^l t^m)a'^\kappa\Big\}$$
$$= -\frac{1}{8\pi^2}\epsilon_{\mu\nu\lambda\kappa}\Big\{tr(t^m t^l t^n)(\alpha k_1^\kappa + (\alpha-\beta)k_2^\kappa)$$
$$- tr(t^n t^l t^m)(\alpha k_2^\kappa + (\alpha-\beta)k_1^\kappa)\Big\}$$

$$\overset{?}{=} \frac{1}{8\pi^2}\epsilon_{\mu\nu\kappa\lambda}\Big\{Tr\big[t_l\{t_m,t_n\}\big]\beta(k_1-k_2)^\kappa - Tr\big[t_l[t_m,t_n]\big](2\alpha+\beta)q^\kappa\Big\}$$

P. 19)

$$iM(\pi^0 \to 2\gamma) = \epsilon_1^\mu \epsilon_2^\nu \Gamma_{\mu\nu}(k_1, k_2, q)$$

with $\Gamma_{\mu\nu}(k_1,k_2,q) = \frac{ie^2}{4\pi^2 f_\pi} \epsilon_{\mu\nu\rho\sigma} k_1^\rho k_2^\sigma$

pion decay constant

Regular Feynman rules to get this matrix element or what's so special about it?

$$\Gamma(\pi^0 \to 2\gamma) = \frac{1}{8\pi} \frac{1}{2M_{\pi^0}} \frac{1}{2} \sum_{\text{pols.}} |M(\pi^0 \to 2\gamma)|^2$$

$$= \frac{1}{32\pi M_{\pi^0}} \sum_{\text{pols.}} |\epsilon_1^\mu \epsilon_2^\nu \Gamma_{\mu\nu}(k_1,k_2,q)|^2$$

Neutral pion decay width = total decay width? → no, only 2γ channel

$$= \frac{1}{32\pi M_{\pi^0}} \sum_{\text{pols.}} \epsilon_1^\mu \epsilon_2^\nu \Gamma_{\mu\nu}(k_1,k_2,q)\, \epsilon_1^{*\alpha} \epsilon_2^{*\beta} \Gamma^*_{\alpha\beta}(k_1,k_2,q)$$

$$= \frac{e^4}{512\pi^5 M_{\pi^0} f_\pi^2} \sum_{\text{pols.}} \epsilon_1^\mu \epsilon_2^\nu \epsilon_1^{*\alpha} \epsilon_2^{*\beta} \left(\epsilon_{\mu\nu\rho\sigma} k_1^\rho k_2^\sigma\right)\left(\epsilon_{\alpha\beta\rho'\sigma'} k_1^{\rho'} k_2^{\sigma'}\right)$$

why factor 1/2? No spins possible? → No, from pion

$$= \frac{e^4}{512\pi^5 M_{\pi^0} f_\pi^2} (-g^{\mu\alpha})(-g^{\nu\beta}) \epsilon_{\mu\nu\rho\sigma} k_1^\rho k_2^\sigma \epsilon_{\alpha\beta\rho'\sigma'} k_1^{\rho'} k_2^{\sigma'}$$

$$= \frac{e^4}{512\pi^5 M_{\pi^0} f_\pi^2} \epsilon_{\mu\nu\rho\sigma} k_1^\rho k_2^\sigma \epsilon^{\mu\nu\rho'\sigma'} k_{1\rho'} k_{2\sigma'}$$

pull one index up → maybe (−) sign?

$$= \frac{e^4}{512\pi^5 M_{\pi^0} f_\pi^2} k_1^\rho k_2^\sigma k_{1\rho'} k_{2\sigma'} \cdot 2\left(\delta_\rho^{\sigma'} \delta_\sigma^{\rho'} - \delta_\rho^{\rho'} \delta_\sigma^{\sigma'}\right)$$

$$= \frac{e^4}{256\pi^5 M_{\pi^0} f_\pi^2} \left(k_1^{\sigma'} k_2^{\rho'} k_{1\rho'} k_{2\sigma'} - k_1^{\rho'} k_2^{\sigma'} k_{1\rho'} k_{2\sigma'}\right)$$

$$= \frac{e^4}{256\pi^5 M_{\pi^0} f_\pi^2} \left((k_1 \cdot k_2)(k_1 \cdot k_2) - k_1^2 k_2^2\right)$$

$$q = k_1 + k_2 \rightsquigarrow M_{\pi^0}^2 = q^2 = (k_1+k_2)^2 = 2k_1k_2$$

$$= \frac{e^4}{256\pi^5 M_{\pi^0} f_\pi^2} \left(\frac{M_{\pi^0}^4}{4}\right)$$

$$\alpha = \frac{e^2}{4\pi} \rightsquigarrow e^4 = 16\pi^2 \alpha^2$$

$$= \frac{16\pi^2\alpha^2}{256\pi^5 M_{\pi^0} f_\pi^2} \frac{M_{\pi^0}^4}{4} = \frac{\alpha^2 M_{\pi^0}^3}{64\pi^3 f_\pi^2}$$