Disclaimer

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23.10.2017 Advanced Quantum theory Exercise 2

Heisenberg picture oper.

$\langle 0 t_b | 0 t_a \rangle^f \,,\; x(t) | 0\, t \rangle = 0$

P2) $L = \frac{m}{2} \dot{x}^2 - \frac{m\omega^2}{2} x^2 + f(t) x \,,\; \omega^2 \to \omega^2 - i\epsilon$

$\partial_t H = 0$

$x_H(t) = e^{iHt} x_S e^{-iHt}$

$x_S |x\rangle_S = x |x\rangle_S$

$|x(t)\rangle_H = e^{iHt} |x\rangle_S$

$x_H(t) |x\rangle_H = x |x\rangle_H$

$|x(t)\rangle_H = |x, t\rangle$

$\langle 0 t_a | 0 t_b \rangle = \langle 0 | e^{-iHt_a} e^{iHt_b} | 0 \rangle$

$\omega^2 - i\epsilon$ only for the int. later? ⇝ not used in a) in tutorial

a) The E.L. eq. yield the e.o.m

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 = m\ddot{x} - (-m\omega^2 x + f(t))$$
$$= m\ddot{x} + m\omega^2 x - f(t)$$

$$\Leftrightarrow m\ddot{x} + m\omega^2 x = f(t)$$

Calculate: $\langle 0 t_b | 0 t_a \rangle^f = \int \mathcal{D}x\, e^{iS[x]}$

First, rewrite $x = x_{cl} + y$, where y is the deviation from the classical path x_{cl}. Now consider

$$L = \frac{1}{2} m \dot{x}^2 - \frac{m\omega^2}{2} x^2 + f(t) x = \frac{1}{2} m (\dot{x}_{cl} + \dot{y})^2 - \frac{m\omega^2}{2}(x_{cl} + y)^2 + f(t)(x_{cl} + y)$$
$$= \frac{1}{2} m \dot{x}_{cl}^2 + \frac{1}{2} m \dot{y}^2 + m \dot{x}_{cl}\dot{y} - \frac{m\omega^2}{2} x_{cl}^2 - \frac{m\omega^2}{2} y^2 - m\omega^2 x_{cl} y + f(t)(x_{cl} + y)$$
$$= \underbrace{\frac{1}{2} m \dot{x}_{cl}^2 - \frac{m\omega^2}{2} x_{cl}^2 + f(t) x_{cl}}_{L_{cl}} + \underbrace{\frac{1}{2} m \dot{y}^2 - \frac{m\omega^2}{2} y^2}_{L_y} + \underbrace{m\dot{x}_{cl}\dot{y} - m\omega^2 x_{cl} y + f(t) y}_{L_m}$$

$$L_m = \frac{d}{dt}(m\dot{x}_{cl} y) - m\ddot{x}_{cl} y - m\omega^2 x_{cl} y + f(t) y$$
$$\overset{e.o.m}{=} \frac{d}{dt}(m\dot{x}_{cl} y) - y(f(t)) + f(t) y = \frac{d}{dt}(m\dot{x}_{cl} y)$$

$$= L_{cl} + L_y + \frac{d}{dt}(m\dot{x}_{cl} y)$$

$$\rightsquigarrow S[x] = \int dt\, L(x, \dot{x}, t) = \int dt\, (L_{cl} + L_y) + \underbrace{\int dt \left\{\frac{d}{dt}(m\dot{x}_{cl} y)\right\}}_{=0,\text{ as } y(t_a) = y(t_b) = 0}$$

$S[x_{cl}], S[y]$? $L_{cl}[x_{cl}], L_y[y]$ ⇝ $S_{cl}[x_{cl}] = \ldots$ $S_y[y] = \ldots$ where $L_{cl}[x_{cl}] = L_f[x_{cl}]$, $L_y[y] = L_0[y]$ and L_f the Lagrangian w/ ext. field

$$\rightsquigarrow \langle 0 t_b | 0 t_a \rangle^f = \int \mathcal{D}x\, e^{iS[x]} \overset{x \mapsto x_{cl} + y}{=} \int \mathcal{D}y\, e^{i(S_{cl} + S_y)}$$

$\mathcal{D}x = \mathcal{D}y$

$x(t_a) = x(t_b) = 0$ $(x(t_a) = x_a, x(t_b) = x_b)$ bzw. $y(t_a) = y(t_b) = 0$ after shift

where $S_{cl} = \int dt\, L_{cl}\,,\; S_y = \int dt\, L_y$

$$\rightsquigarrow \langle 0 t_b | 0 t_a \rangle^f = e^{iS_{cl}} \underbrace{\int \mathcal{D}y\, e^{iS_y}}$$

$$\rightsquigarrow \frac{\langle 0 t_b | 0 t_a \rangle^f}{\langle 0 t_b | 0 t_a \rangle^{f=0}} = e^{iS_{cl}}$$

$\langle 0 t_b | 0 t_a \rangle^{f=0}$ with $y \mapsto x$

5) The solution to x_{cl} is given by $x_{cl}(t) = \int dt'\, G(t-t')\, f(t')$ ← Green's fct. of L_{tt}

e.o.m.: $m\ddot{x}_{cl} + m\omega^2 x_{cl} = f(t) \Rightarrow \underbrace{m\left(\frac{d^2}{dt^2}+\omega^2\right)}_{=:L_{tt}} x_{cl} = f(t)$

The greens fct is given by $L_{tt}\, G(t-t') = \delta(t-t') \overset{\text{repr. of } \delta}{=} \int \frac{dE}{2\pi} e^{-iE(t-t')}$

Additionally, we can write $G(t-t') = \int \frac{dE}{2\pi} e^{-iE(t-t')}\, \tilde{G}(E)$ ← Fourier trafo of $G(t-t')$

$$\Rightarrow m\left(\frac{d^2}{dt^2}+\omega^2\right)\int \frac{dE}{2\pi} e^{-iE(t-t')}\tilde{G}(E) = \int \frac{dE}{2\pi} e^{-iE(t-t')}$$

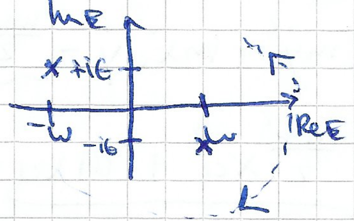
$$\Leftrightarrow \int \frac{dE}{2\pi}\, m\,(-E^2+\omega^2)\, e^{-iE(t-t')}\tilde{G}(E) = \int \frac{dE}{2\pi} e^{-iE(t-t')}$$

$$\Rightarrow \tilde{G}(E) = \frac{1}{m(\omega^2-E^2)}$$

$$\rightsquigarrow G(t-t') = \int \frac{dE}{2\pi} e^{-iE(t-t')}\frac{1}{m(\omega^2-E^2)} \overset{\omega^2\to\omega^2-i\epsilon}{=} \int \frac{dE}{2\pi}\, \frac{1}{m}\,\overbrace{e^{-iE(t-t')}\,\frac{1}{(E-\omega+i\epsilon)(E+\omega-i\epsilon)}}^{h(E)}$$

where we used $\omega^2 - i\epsilon = E^2 \Leftrightarrow E = \pm\sqrt{\omega^2-i\epsilon} \approx \pm(\omega - i\epsilon)$

Sketch of the poles:



We use the residue theorem to calculate the integral.

For $t-t'>0$: Close the integration in lower arc, in order to achieve an exponential decay:

$$G(t-t') = \frac{1}{2\pi m}(-2\pi i)\,\mathrm{Res}(h, E=\omega-i\epsilon)$$

$$\mathrm{Res}(h, E=\omega-i\epsilon) = e^{-iE(t-t')}\frac{1}{E+\omega-i\epsilon}\Big|_{E=\omega-i\epsilon} = \frac{e^{-i(t-t')(\omega-i\epsilon)}}{2\omega-2i\epsilon} \xrightarrow{\epsilon\to 0} \frac{e^{-i\omega(t-t')}}{2\omega}$$

$$= -\frac{i}{2m\omega} e^{-i\omega(t-t')}$$

For $t-t'<0$: upper arc: $G(t-t') = \frac{1}{2\pi m}(2\pi i)\,\mathrm{Res}(h, E=-\omega+i\epsilon)$

$$= -\frac{i}{2m\omega} e^{i\omega(t-t')}$$

$$\rightsquigarrow G(t-t') = \theta(t-t')\left\{-\frac{i}{2m\omega}e^{-i\omega(t-t')}\right\} + \theta(t'-t)\left\{-\frac{i}{2m\omega}e^{i\omega(t-t')}\right\}$$

$L_x u(x) = f(x)$?, we know $L_x G(x,s) = \delta(x-s)$ | $\int ds$ and $\times f(s)$

$\rightsquigarrow \int ds\, L_x G(x,s) f(s) = \int ds\, \delta(x-s) f(s)$

$\rightsquigarrow L_x \int ds\, G(x,s) f(s) = f(x)$ $\rightsquigarrow \int ds\, G(x,s) f(s)$ is the solution

transition ampl. indep. of x_a, x_b?
$\rightsquigarrow$ not like QFT $\phi(\vec{x},t)$

Why only integrate w/ respect to t
$\rightsquigarrow$ only t dimension in / no spatial in Operator

Already assume (above) here that $G(t,t') = G(t-t')$?
Fourier trafo still just integrate $\vec{E}$?
$\rightsquigarrow$ in tut. yes

Factor of m on which side
$\rightsquigarrow$ doesn't matter
on left in tut

$\pm i\epsilon$? ← can be derived (non-trivial)
$\rightsquigarrow$ take care of right factors

Why is $G(t-t')$ the propagator and not the convolution x_{cl}?

What are we doing in this exercise?

P3) $\hat{H} = \frac{\hbar^2 \hat{k}^2}{2m} + V(\hat{x})$, from now on, $\hbar = 1 \rightsquigarrow \hat{H} = \frac{\hat{k}^2}{2m} + V(\hat{x})$

a) $\langle x_b t_b | x_a t_a \rangle = \langle x_b | e^{-iH(t_b - t_a)} | x_a \rangle \overset{\text{Baker-Hausdorff-Campbell}}{=} \langle x_b | e^{-iHt_b} e^{-iH(t'-t')} e^{iHt_a} | x_a \rangle$

Semi group property?

$t_b > t' > t_a$

$$= \int dx' \langle x_b | e^{-iH(t_b - t')} | x' \rangle \langle x' | e^{-iH(t'-t_a)} | x_a \rangle$$

$$= \int dx' \langle x_b t_b | x' t' \rangle \langle x' t' | x_a t_a \rangle$$

Why $t_b > t' > t_a$?

b) $\langle x_b t_b | = \langle x, t + \Delta t |$, $t = t'$

Why $t = t'$? $t = t' \rightsquigarrow x = x'$?

$$\rightsquigarrow \langle x, t + \Delta t | x_a t_a \rangle = \int dx' \langle x, t + \Delta t | x' t \rangle \langle x' t | x_a t_a \rangle$$

$$\Leftrightarrow \langle x, t | x_a t_a \rangle + \frac{d}{dt} \langle x, t + \Delta t | x_a t_a \rangle \Big|_{t=0} \Delta t + \mathcal{O}((\Delta t)^2)$$

$\frac{d}{dt} \langle x, t + \Delta t | \ldots$?

$$= \int dx' \langle x | e^{-iH\Delta t} | x' \rangle \langle x' t | x_a t_a \rangle$$

$$= \int dx' \langle x | \mathbb{1} - iH\Delta t + \ldots | x' \rangle \langle x' t | x_a t_a \rangle$$

$H = \frac{\hat{k}^2}{2m} + V(\hat{x})$

$\int \frac{dk}{2\pi}$ from $|k\rangle\langle k|$ and! δ go from $-\infty$ to ∞?

$$= \int dx' \left\{ \delta(x - x') - iV(x) \Delta t \, \delta(x - x') - i\Delta t \int \frac{dk}{2\pi} \langle x | \frac{\hat{k}^2}{2m} | k \rangle \langle k | x' \rangle + \ldots \right\} \langle x' t | x_a t_a \rangle$$

$$= \int dx' \left\{ \int \frac{dk}{2\pi} e^{-ik(x'-x)} - iV(x) \Delta t \int \frac{dk}{2\pi} e^{-ik(x'-x)} - i\Delta t \int \frac{dk}{2\pi} \frac{k^2}{2m} e^{-ik(x'-x)} + \ldots \right\} \langle x' t | x_a t_a \rangle$$

$$= \int dx' \int \frac{dk}{2\pi} e^{-ik(x'-x)} \left\{ \mathbb{1} - iV(x) \Delta t - i \frac{k^2}{2m} \Delta t + \ldots \right\} \langle x' t | x_a t_a \rangle$$

What about factors like $V(x) k^2 V(x) k^2$ ...? $\rightsquigarrow$ are $\mathcal{O}((\Delta t)^2)$ thus don't regard them ...

$$= \int dx' \int \frac{dk}{2\pi} e^{-ik(x'-x)} \left\{ \mathbb{1} - i\Delta t \left(\frac{k^2}{2m} + V(x) \right) + \ldots \right\} \langle x' t | x_a t_a \rangle$$

$$= \int dx' \int \frac{dk}{2\pi} e^{-ik(x'-x)} e^{-i\Delta t \left(\frac{k^2}{2m} + V(x) \right)} \langle x' t | x_a t_a \rangle$$

$$= \int dx' \int \frac{dk}{2\pi} e^{-\frac{i\Delta t}{2m} k^2 - i(x'-x)k} e^{-i\Delta t V(x)} \langle x' t | x_a t_a \rangle$$

$$= \int dx' \frac{e^{-i\Delta t V(x)}}{2\pi} \sqrt{\frac{2\pi m}{i\Delta t}} e^{-\frac{(x'-x)^2 m}{2i\Delta t}} \langle x' t | x_a t_a \rangle$$

$$= \int dx' \sqrt{\frac{m}{2\pi i \Delta t}} e^{-i\Delta t V(x)} e^{-\frac{(x'-x)^2 m}{2i\Delta t}} \langle x' t | x_a t_a \rangle$$

$$= \underbrace{\int dx' \sqrt{\frac{m}{2\pi i \Delta t}} e^{-i\Delta t \left(V(x) - \frac{m}{2} \left(\frac{x'-x}{\Delta t} \right)^2 \right)} \langle x' t | x_a t_a \rangle}_{\equiv \text{RHS}}$$

Wofür diese Umschreibung?

Why big phases?

c) Big phases in the exponential might cancel each other out. We thus have the main contribution for $(x'-x)\approx 0$.

derivative w/ respect to x'?

$$\langle x' t | x_a t_a\rangle = \langle x t | x_a t_a\rangle + \frac{d}{dx'}\langle x' t | x_a t_a\rangle\Big|_{x'=x}(x'-x) + \frac{1}{2}\frac{d^2}{dx'^2}\langle x' t | x_a t_a\rangle\Big|_{x'=x}(x'-x)^2 + \mathcal{O}((x'-x)^2)$$

$$\rightsquigarrow \text{RHS} = \int dx' \sqrt{\frac{m}{2\pi i \Delta t}}\, e^{-i\Delta t V(x)}\, e^{\frac{im}{2\Delta t}(x'-x)^2}\langle x' t | x_a t_a\rangle$$

Taylor exp as well?

$$= \int dx' \sqrt{\frac{m}{2\pi i \Delta t}}\left(1 - i\Delta t V(x) + \mathcal{O}((\Delta t)^2)\right) e^{-\frac{m}{2i\Delta t}(x'-x)^2}$$
$$\times\left(\langle x t | x_a t_a\rangle + \frac{d}{dx}\langle x t | x_a t_a\rangle (x'-x) + \frac{1}{2}\frac{d^2}{dx^2}\langle x t | x_a t_a\rangle (x'-x)^2 + \dots\right)$$

Integral $\int e^{x^2}\cdot x$ wie gelöst?

$x'-x=\tilde{x}$ and antisym. term in $x'-x=\tilde{x}$

$$\overset{}{=}\int d\tilde{x} \sqrt{\frac{m}{2\pi i \Delta t}}\left(1 - i\Delta t V(x) + \mathcal{O}((\Delta t)^2)\right) e^{-\frac{m}{2i\Delta t}\tilde{x}^2}$$
$$\times\left(\langle x t | x_a t_a\rangle + \frac{1}{2}\frac{d^2}{dx^2}\langle x t | x_a t_a\rangle \tilde{x}^2 + \dots\right)$$

$\mathcal{O}((x'-x)^2) \mapsto \mathcal{O}(\tilde{x}^2)$ during substitution? No dependence left after integrating?

$$= \left(1 - i\Delta t V(x) + \mathcal{O}((\Delta t)^2)\right)\sqrt{\frac{m}{2\pi i \Delta t}}\left\{\int d\tilde{x}\, e^{-\frac{m}{2i\Delta t}\tilde{x}^2}\langle x t | x_a t_a\rangle + \frac{1}{2}\frac{d^2}{dx^2}\langle x t | x_a t_a\rangle \int d\tilde{x}\, e^{-\frac{m}{2i\Delta t}\tilde{x}^2}\tilde{x}^2 + \dots\right\}$$

$$= \left(1 - i\Delta t V(x) + \mathcal{O}((\Delta t)^2)\right)\sqrt{\frac{m}{2\pi i \Delta t}}\left\{\sqrt{\frac{2\pi i \Delta t}{m}}\langle x t | x_a t_a\rangle + \frac{1}{2}\frac{d^2}{dx^2}\langle x t | x_a t_a\rangle \frac{\sqrt{2\pi}}{m^{3/2}}(i\Delta t)^{3/2} + \dots\right\}$$

$$= \left(1 - i\Delta t V(x) + \mathcal{O}((\Delta t)^2)\right)\left\{\langle x t | x_a t_a\rangle + \frac{i\Delta t}{2m}\frac{d^2}{dx^2}\langle x t | x_a t_a\rangle + \dots\right\}$$

Why partial derivatives?

b) $\underset{\text{LHS}}{\rightsquigarrow}$ $\langle x, t | x_a t_a\rangle + \frac{d}{dt}\langle x, t | x_a t_a\rangle \Delta t + \dots$

$$= \langle x, t | x_a t_a\rangle - i\Delta t V(x)\langle x, t | x_a t_a\rangle + \frac{i\Delta t}{2m}\frac{d^2}{dx^2}\langle x t | x_a t_a\rangle + \frac{1}{2m}(\Delta t)^2 V(x)\frac{d^2}{dx^2}\langle x t | x_a t_a\rangle + \dots$$

S.Eq. for wave fct? Not done?

$$\rightsquigarrow i\frac{d}{dt}\langle x t, x_a t_a\rangle = -\frac{1}{2m}\frac{d^2}{dx^2}\langle x t | x_a t_a\rangle + V(x)\langle x t | x_a t_a\rangle$$