Disclaimer

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https://www.physics-and-stuff.com/

I raise no claim to correctness and completeness of the given solutions! This equally applies to the corrections mentioned above.

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Advanced Quantum Field Theory Exercise 4

P(6) a) $\phi(x) = \frac{\delta}{\delta j(x)} E[j]$, $\Gamma[\phi] = E[j[\phi]] - \int d^4x\, j(x)\phi(x)$

How to invert $\phi = \phi[j]$ for $j = j[\phi]$? ⇝ basically possibly (has not seen it)

$$\frac{\delta \Gamma[\phi]}{\delta \phi(x)} = \int d^4z \frac{\delta j[\phi](z)}{\delta \phi(x)} \underbrace{\frac{\delta E[j[\phi]]}{\delta j[\phi](z)}}_{=\phi(z)} - \int d^4y \frac{\delta j[\phi](y)}{\delta \phi(x)} \phi(y) - j(x)$$

j still depends on ϕ, so have to write $j[\phi](z)$ etc.? ⇝ w/o dependance in class

$$= 0 - j(x) = -j(x)$$

b) $iE[j] = \sum_n \frac{i^n}{n!} \int d^4x_1 \dots d^4x_n\, G_c(x_1, \dots, x_n)\, j(x_1) \dots j(x_n)$

$\Gamma[\phi] = -\sum_n \frac{(-i)^n}{n!} \int d^4x_1 \dots d^4x_n\, \Gamma(x_1, \dots, x_n)\, \phi(x_1) \dots \phi(x_n)$

Where from $\Gamma[\phi]$? ⇝ formal taylor exp. (for functionals though!)

$$\frac{\delta \phi(x)}{\delta \phi(y)} = \delta(x-y)$$

$$\frac{\delta \phi(x)}{\delta \phi(y)} = \frac{\delta}{\delta \phi(y)}\left(\frac{\delta}{\delta j(x)} E[j]\right) = \int d^4z \frac{\delta j(z)}{\delta \phi(y)} \frac{\delta^2 E[j]}{\delta j(z)\delta j(x)}$$

Could also use chain rule w/ respect to $\frac{\delta E}{\delta \phi}\frac{\delta \phi(z)}{\delta j(x)}$ ⇝ different result but same in the end

One quickly finds

$$\left.\frac{\delta j(z)}{\delta \phi(y)}\right|_{j=\phi=0} \overset{a)}{=} -\left.\frac{\delta^2 \Gamma[\phi]}{\delta \phi(y)\delta \phi(z)}\right|_{j=\phi=0} = -\Gamma(y,z),$$ where we used that the n derivatives can be distributed in $n!$ ways and thus cancelling the $\frac{1}{n!}$.

Only if $\phi = 0 = j$ in the end valid? ⇝ set it to zero in the end.

Furthermore,

$$\left.\frac{\delta^2 E[j]}{\delta j(z)\delta j(x)}\right|_{j=\phi=0} = \frac{1}{i}\left(-G_c(z,x)\right)$$

$\Gamma(y,z) = \Gamma(y-z)$ → transl. inv.? or why sym. ⇝ sym. as. def. is sym. in int. change of $\phi(x_i)$'s

$$\left.\frac{\delta \phi(x)}{\delta \phi(y)}\right|_{j=\phi=0} = -i \int d^4z\, G_c(x-z)\, \Gamma(y-z)$$

$$= \delta(x-y)$$

We thus have

$$i\delta(x-y) = \int d^4z\, G_c(x-z)\, \Gamma(y-z)$$

Using the Fourier decomposition:

$$G_c(x-z) = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-z)} \tilde{G}_c(p)$$

$$\Gamma(y-z) = \int \frac{d^4p'}{(2\pi)^4} e^{-ip'(y-z)} \tilde{\Gamma}(p')$$

$$\delta(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)}$$, we find

$$\int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)}\, i = \int d^4z \frac{d^4p\, d^4p'}{(2\pi)^4 (2\pi)^4} e^{-ip(x-z)} e^{-ip'(z-y)} \tilde{G}_c(p)\, \tilde{\Gamma}(p')$$

Can we just do $p' \to -p'$? Problem $\tilde{\Gamma}(-p')$? $\rightsquigarrow$ transl. inv.!

$$p' \mapsto -p' \quad = \int d^4z \frac{d^4p\, d^4p'}{(2\pi)^4 (2\pi)^4} e^{-iz(p'-p)} e^{-ipx} e^{ip'y} \tilde{G}_c(p)\, \tilde{\Gamma}(p')$$

$$= \int \frac{d^4p\, d^4p'}{(2\pi)^4} \delta(p'-p) e^{-ipx} e^{ip'y} \tilde{G}_c(p)\, \tilde{\Gamma}(p')$$

$$= \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \tilde{G}_c(p)\, \tilde{\Gamma}(p)$$

„Koeff. vgl."? Could also be true in diff. case? $\rightsquigarrow$ different way in tut.

$$\rightsquigarrow\ i = \tilde{G}_c(p)\, \tilde{\Gamma}(p)$$

c) We had $\delta(x-y) = \int d^4z \, \frac{\delta j(z)}{\delta \phi(y)} \frac{\delta^2 E[j]}{\delta j(z)\delta j(x)}$

$\frac{\delta}{\delta\phi(w)}$ $$0 = \int d^4z \, \frac{\delta^2 j(z)}{\delta\phi(w)\delta\phi(y)} \frac{\delta^2 E[j]}{\delta j(z)\delta j(x)} + \int d^4z \, d^4v \, \frac{\delta j(v)}{\delta\phi(w)} \frac{\delta j(z)}{\delta\phi(y)} \frac{\delta^3 E[j]}{\delta j(v)\delta j(z)\delta j(x)}$$

Why derive w/ respect to $\phi(w)$ and not $j(w)$? ⇝ gives us what we want.

Again, looking at the limits $\phi=0$ and $j=0$ after performing the derivations, we find:

$$\frac{\delta^2 j(z)}{\delta\phi(w)\delta\phi(y)} = -\frac{\delta^3\Gamma[\phi]}{\delta\phi(w)\delta\phi(y)\delta\phi(z)} = i\Gamma(w,y,z)$$

$$\frac{\delta^2 E[j]}{\delta j(z)\delta j(x)} = -\frac{1}{i} G_c(x-z)$$

$$\frac{\delta j(i)}{\delta\phi(j)} = -\frac{\delta^2\Gamma[\phi]}{\delta\phi(j)\delta\phi(i)} = -\Gamma(j,i) = -\Gamma(j-i)$$

$$\frac{\delta^3 E[j]}{\delta j(v)\delta j(z)\delta j(x)} = \frac{1}{i}\left(-i\, G_c(v,z,x)\right) = -G_c(v,z,x)$$

For 3 vars, not just $G_c(x-y-z)$ anymore? Still sym.? Still sym. & transl. inv. but does not only dep. on diff.

$$\leadsto 0 = -\int d^4z \, G_c(x-z)\Gamma(w,y,z) - \int d^4z \, d^4v \, \Gamma(v-w)\Gamma(z-y) G_c(v,z,x)$$

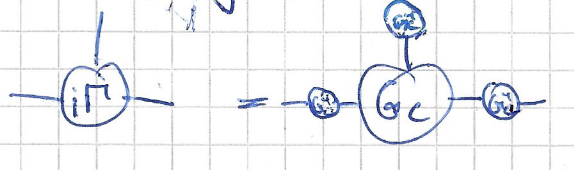
$$\Longleftrightarrow \int d^4z \, G_c(x-z)\Gamma(w,y,z) = -\int d^4z \, d^4v \, \Gamma(v-w)\Gamma(z-y)G_c(v,z,x)$$

Multiplying w/ $\int d^4x \, \Gamma(u,x)$ and using $\delta(x-y) = -i\int d^4z \, G_c(x-z)\Gamma(y-z)$

Multiplying w/ integral dep. of sth. which we already have a function of? ⇝ holds $\forall x$?!

$$\leadsto \int d^4z \, d^4x \, G_c(x-z)\Gamma(u-x)\Gamma(w,y,z) = -\int d^4z \, d^4v \, d^4x \, \Gamma(v-w)\Gamma(z-y)\Gamma(u-x)G_c(v,z,x)$$

$$\Longleftrightarrow +i\,\Gamma(w,y,u) = -\int d^4z \, d^4v \, d^4x \, G_c(v,z,x)\Gamma(v-w)\Gamma(z-y)\Gamma(u-x)$$

⇝ 

as $\Gamma(p) \propto \tilde{G}_c^{-1}(p)$