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20.11.2017 **Advanced Quantum Field Theory Ex. 6** Marvin Zanke

P8) a) $[k^2-m^2]-[(k-p)^2-m^2]+p^2$

$$= k^2-m^2-k^2-p^2+2kp+m^2+p^2 = 2kp$$

Calculate

$$2p_\mu \int \frac{d^dk}{(2\pi)^d} \frac{-ik^\mu}{[k^2-m^2+i\epsilon][(k-p)^2-m^2+i\epsilon]} = -i\int \frac{d^dk}{(2\pi)^d} \frac{[k^2-m^2]-[(k-p)^2-m^2]+p^2}{[k^2-m^2+i\epsilon][(k-p)^2-m^2+i\epsilon]}$$

$$= \int \frac{d^dk}{(2\pi)^d} \frac{-i}{(k-p)^2-m^2+i\epsilon} - \int \frac{d^dk}{(2\pi)^d} \frac{-i}{k^2-m^2+i\epsilon} + p^2 \int \frac{d^dk}{(2\pi)^d} \frac{-i}{[k^2-m^2+i\epsilon][(k-p)^2-m^2+i\epsilon]}$$

$k'=k-p$

$$= \int \frac{d^dk}{(2\pi)^d} \frac{-i}{k'^2-m^2+i\epsilon} - \int \frac{d^dk}{(2\pi)^d} \frac{-i}{k^2-m^2+i\epsilon} + p^2 \int \frac{d^dk}{(2\pi)^d} \frac{-i}{[k^2-m^2+i\epsilon][(k-p)^2-m^2+i\epsilon]}$$

$$= p^2 \underbrace{\int \frac{d^dk}{(2\pi)^d} \frac{-i}{[k^2-m^2+i\epsilon][(k-p)^2-m^2+i\epsilon]}}_{= \frac{1}{16\pi^2} B_0(p^2,m,m)}$$

formal way? multiply p_μ on both sides? why possible? → see right

$$\Longleftrightarrow 2p_\mu \int \frac{d^dk}{(2\pi)^d} \frac{-ik^\mu}{[k^2-m^2+i\epsilon][(k-p)^2-m^2+i\epsilon]}$$

$$= p_\mu p^\mu \int \frac{d^dk}{(2\pi)^d} \frac{-i}{[k^2-m^2+i\epsilon][(k-p)^2-m^2+i\epsilon]} = p_\mu \frac{p^\mu}{16\pi^2} B_0(p^2,m,m)$$

Why factor out $1/16\pi^2$? → no deeper meaning

$$\rightsquigarrow \int \frac{d^dk}{(2\pi)^d} \frac{-ik^\mu}{[k^2-m^2+i\epsilon][(k-p)^2-m^2+i\epsilon]} = \frac{p^\mu}{32\pi^2} B_0(p^2,m,m)$$

$$\int \frac{d^dk}{(2\pi)^d} \frac{k^\mu}{\dots} = p^\mu f(p^2,m^2) \quad \text{only avail. } \frac{1}{\dots}$$

$$2p_\mu p^\mu f(p^2,m^2) = p_\mu p^\mu \underbrace{\int \frac{d^dk}{(2\pi)^d} \frac{1}{\dots}}_{\frac{i}{16\pi^2} B_0(p^2\dots}$$

b) $$\int \frac{d^dk}{(2\pi)^d} \frac{-i}{[k^2-m^2+i\epsilon][(k-p)^2-m^2+i\epsilon]}$$

Feynman Par.

$$= -i \int \frac{d^dk}{(2\pi)^d} \int_0^1 dx \frac{1}{\underbrace{\{(1-x)[k^2-m^2+i\epsilon]+x[(k-p)^2-m^2+i\epsilon]\}}_{=D}{}^2}$$

What about the $+i\epsilon$? → after Wick rotation, the poles aren't hit anymore in integration → set $\epsilon \to 0$

$$D = k^2-m^2+i\epsilon - xk^2 + m^2x - ix\epsilon + xk^2 + xp^2 - 2pkx - m^2x + i\epsilon x$$

$$= k^2-m^2+xp^2-2pkx+i\epsilon = (k-px)^2 - p^2x^2 + xp^2 - m^2 + i\epsilon$$

$$= (k-px)^2 - [x(x-1)p^2+m^2] + i\epsilon$$

$$\overset{k'=k-px}{=} -i\int\frac{d^d k'}{(2\pi)^d}\int_0^1 dx\,\frac{1}{[k'^2-\Delta+i\epsilon]^2}\,,\quad \Delta=[x(x-1)p^2+m^2]$$

Wick rot. $k^0=ik_E^0$, $k^i=k_E^i$

$$= -(i)^2\int\frac{d^d k_E}{(2\pi)^d}\int_0^1 dx\,\frac{1}{[k_E^2+\Delta-i\epsilon]^2} = \int\frac{d^d k_E}{(2\pi)^d}\int_0^1 dx\,\frac{1}{[k_E^2+\Delta-i\epsilon]^2}$$

$$=\int_0^1 dx\,\frac{1}{(4\pi)^{d/2}}\,\frac{\Gamma(2-d/2)}{\Gamma(2)}\,\Delta^{d/2-2}$$

$$\overset{d=4-2\epsilon}{=}\int_0^1 dx\,\frac{1}{(4\pi)^{2-\epsilon}}\,\Gamma(\epsilon)\,\Delta^{-\epsilon} = \frac{1}{16\pi^2}\,\Gamma(\epsilon)\int_0^1 dx\left(\frac{4\pi}{\Delta}\right)^{\epsilon}$$

go to 4 dim. here? And why 2ϵ?, we can set it to any $-k\cdot\epsilon$ we want; yields the same in the end

$$=\frac{1}{16\pi^2}\left(\frac{1}{\epsilon}-\gamma_E+\mathcal{O}(\epsilon)\right)\int_0^1 dx\left(\frac{4\pi}{\Delta}\right)^{\epsilon}$$

$x^\epsilon = e^{\log x\cdot\epsilon} = 1+\epsilon\log x+\mathcal{O}(\epsilon^2)$

$$=\frac{1}{16\pi^2}\left(\frac{1}{\epsilon}-\gamma_E+\mathcal{O}(\epsilon)\right)\int_0^1 dx\left(1+\epsilon\log\left(\frac{4\pi}{\Delta}\right)+\mathcal{O}(\epsilon^2)\right)$$

$$=\frac{1}{16\pi^2}\int_0^1 dx\left\{\frac{1}{\epsilon}+\log\left(\frac{4\pi}{\Delta}\right)-\gamma_E+\mathcal{O}(\epsilon)\right\}$$

$$=\frac{1}{16\pi^2}\int_0^1 dx\left\{\frac{1}{\epsilon}-\gamma_E+\log(4\pi)-\log(\Delta)+\mathcal{O}(\epsilon)\right\}$$

c) $\displaystyle\int_0^1 dx\,\log\Delta = \int_0^1 dx\,\log(x(x-1)p^2+m^2)$

$|\,x(x-1)p^2+m^2 = x^2p^2-xp^2+m^2 = (x-1/2)^2p^2-p^2/4+m^2$

$$=\int_{-1/2}^{1/2} dy\,\log(y^2p^2-p^2/4+m^2)\cdot 1$$

$$= y\log(y^2p^2-p^2/4+m^2)\Big|_{-1/2}^{1/2}-\int_{-1/2}^{1/2}dy\,y\,\frac{2p^2y}{y^2p^2-p^2/4+m^2}$$

$$=\tfrac{1}{2}\log(m^2)+\tfrac{1}{2}\log(m^2)-2\int_{-1/2}^{1/2}dy\,\frac{p^2y^2}{y^2p^2-p^2/4+m^2}$$

$z=py$, $\frac{dz}{dy}=p$

$$=\log(m^2)-\frac{2}{p}\int_{-p/2}^{p/2}dz\,\frac{z^2+a-a}{z^2+a}\,,\quad a=m^2-\frac{p^2}{4}$$

$w=\frac{z}{\sqrt{a}}$, $\frac{dw}{dz}=\frac{1}{\sqrt{a}}$

$$=\log(m^2)-\frac{2}{p}\left(\int_{-p/2}^{p/2}dy\,1-a\frac{\sqrt{a}}{a}\int_{-p/2\sqrt{a}}^{p/2\sqrt{a}}dw\,\frac{1}{w^2+1}\right)$$

$$= \log(m^2) - 2 + \frac{2}{p} a^{1/2} \int_{-\frac{p}{2\sqrt{a}}}^{p/2\sqrt{a}} dw \, \frac{1}{w^2+1}$$

$$= \log(m^2) - 2 + \frac{2}{p} a^{1/2} \arctan w \Big|_{-p/2\sqrt{a}}^{p/2\sqrt{a}}$$

$$= \log(m^2) - 2 + \frac{2}{p} a^{1/2} \left(2 \arctan\left(\frac{p}{2\sqrt{a}}\right)\right)$$

$$\left| \quad \arctan z = \int_0^z dt \, \frac{1}{1+t^2} = \int_0^z dt \, \frac{1}{2}\left(\frac{(1+it)+(1-it)}{(1+it)(1-it)}\right) \right.$$

$$= \frac{1}{2}\left(\int_0^z dt \, \frac{1}{1-it} + \int_0^z dt \, \frac{1}{1+it}\right) = \frac{1}{2i}\left(\log(1+iz) - \log(1-iz)\right)$$

$$= \log(m^2) - 2 + \frac{4\sqrt{a}}{p} \frac{1}{2i}\left(\log\left(1 + i\frac{p}{2\sqrt{a}}\right) - \log\left(1 - i\frac{p}{2\sqrt{a}}\right)\right)$$

Why $p^2<0$? What if $p^2>0$? ⇝ see tut.

$$= \log(m^2) - 2 - i\frac{2\sqrt{a}}{p}\left(\log\left(1 + i\frac{p}{2\sqrt{a}}\right) - \log\left(1 - i\frac{p}{2\sqrt{a}}\right)\right)$$

$$\left| \quad \sqrt{a} = \sqrt{m^2 - p^2/4} = i\sqrt{p^2/4 - m^2} = i\,p/2\sqrt{1 - \frac{4m^2}{p^2}} = i\,p/2\,\sigma \right.$$

But $p^2>4m^2$ still?

$$= \log(m^2) - 2 + \sigma\left(\log\left(1 + \frac{1}{\sigma}\right) - \log\left(1 - \frac{1}{\sigma}\right)\right)$$

$$= \log(m^2) - 2 + \sigma \log\left(\frac{\frac{\sigma+1}{\sigma}}{\frac{\sigma-1}{\sigma}}\right) = \log(m^2) - 2 + \sigma \log\frac{\sigma+1}{\sigma-1}$$

Why Im part only if onshell?

d) $$B_0(p^2,m,m) = \frac{1}{\pi}\int_{4m^2}^{\infty} dz \, \frac{\operatorname{Im} B_0(z,m,m)}{z-p^2}$$

$$B_0(0,m,m) = \frac{1}{\pi}\int_{4m^2}^{\infty} dz \, \frac{\operatorname{Im} B_0(z,m,m)}{z}$$

Where from $\operatorname{Im} B_0 = \pi\theta(\ldots)\sqrt{\ldots}$? Usually $\operatorname{Im} M = \lambda^2/2 \; 1/16\pi^2 \sqrt{1-\frac{4m^2}{s}}$ ⇝ factors not considered / already factored out

$$\leadsto B_0(p^2,m,m) - B_0(0,m,m) = \frac{1}{\pi}\int_{4m^2}^{\infty} dz \left\{\frac{\operatorname{Im} B_0(z,m,m)}{z-p^2} - \frac{\operatorname{Im} B_0(z,m,m)}{z}\right\}$$

$$= \frac{1}{\pi}\int_{4m^2}^{\infty} dz \, \frac{\operatorname{Im} B_0(z,m,m)\, p^2}{(z-p^2)\, z} = \frac{p^2}{\pi}\int_{4m^2}^{\infty} \frac{dz}{z} \, \frac{\operatorname{Im} B_0(z,m,m)}{(z-p^2)}$$

Once subtracted disp. rel. usually derived differently? ⇝ better other way (see tut.) as 1st int. not conv.

Now, we want to calculate this w/ $\operatorname{Im} B_0(p^2,m,m) = \pi\,\theta(p^2-4m^2)\sqrt{1-\frac{4m^2}{p^2}}$

$$\rightsquigarrow \frac{p^2}{\pi}\int_{4m^2}^{\infty}\frac{dz}{z}\,\frac{\pi\sqrt{1-\frac{4m^2}{z}}}{(z-p^2)} = p^2\int_{4m^2}^{\infty}\frac{dz}{z}\,\frac{\sqrt{1-\frac{4m^2}{z}}}{z-p^2}$$

$$\xi=\sqrt{1-\frac{4m^2}{z}} \rightsquigarrow z=\frac{4m^2}{1-\xi^2}\,,\quad \frac{d\xi}{dz}=\frac{1}{2\xi}\,\frac{4m^2}{z^2}$$

$$\sigma=\sqrt{1-\frac{4m^2}{p^2}} \rightsquigarrow p^2=\frac{4m^2}{1-\sigma^2}$$

$$=\frac{4m^2}{1-\sigma^2}\int_0^1 d\xi\,(2\xi)\left(\frac{z^2}{4m^2}\right)\frac{\xi}{\frac{4m^2}{1-\xi^2}-\frac{4m^2}{1-\sigma^2}}$$

$$=\frac{4m^2}{1-\sigma^2}\int_0^1 d\xi\,(2\xi)\,\frac{4m^2}{1-\xi^2}\,\frac{1}{(4m^2)^2}\,\frac{\xi}{\frac{1}{1-\xi^2}-\frac{1}{1-\sigma^2}}$$

$$=\frac{2}{1-\sigma^2}\int_0^1 d\xi\,\frac{\xi^2}{1-\frac{1-\xi^2}{1-\sigma^2}} = \frac{2}{1-\sigma^2}\int_0^1 d\xi\,\frac{\xi^2}{\frac{(1-\sigma^2)-(1-\xi^2)}{1-\sigma^2}}$$

$$=2\int_0^1 d\xi\,\frac{\xi^2-\sigma^2+\sigma^2}{\xi^2-\sigma^2} = 2\left(\int_0^1 d\xi\,1+\int_0^1 d\xi\,\frac{\sigma^2}{\xi^2-\sigma^2}\right)$$

$$=2+2\sigma^2\int_0^1 d\xi\,\frac{1}{\xi^2-\sigma^2} \overset{k=\xi/\sigma}{=} 2+2\sigma\int_0^{1/\sigma}dk\,\frac{1}{k^2-1}$$

$$\int dk\,\frac{1}{k^2-1}=\frac12\int dk\,\frac{(1-k)+(1+k)}{(k+1)(k-1)}=\frac12\int dk\left(\frac{1}{k-1}-\frac{1}{k+1}\right)$$

$$=\frac12\left(\log|k-1|-\log|k+1|\right)$$

$$=2+2\sigma\,\frac12\left(\log|k-1|-\log|k+1|\right)\Big|_0^{1/\sigma} = 2+\sigma\log\left|\frac{1/\sigma-1}{1/\sigma+1}\right|$$

$$=2+\sigma\log\left|\frac{\frac{1-\sigma}{\sigma}}{\frac{1+\sigma}{\sigma}}\right| = 2+\sigma\log\left|\frac{\sigma-1}{\sigma+1}\right| = 2+\sigma\log\frac{\sigma-1}{\sigma+1}$$

$\sigma=\sqrt{1-\frac{4m^2}{p^2}}>1$ for $p^2<0$

$\log|x|=\int 1/x$? Why $k>1$? $\rightsquigarrow$ or make x always positive

$\rightsquigarrow$ in c), we had $\int_0^1 dx\log\Delta = -2+\log m^2+\sigma\log\frac{\sigma+1}{\sigma-1}$, which implies for b),

$$B_0(p^2;m,m)=\frac{1}{16\pi^2}\int_0^1 dx\left(\frac1\epsilon-\gamma_E+\log 4\pi+\mathcal{O}(\epsilon)\right)+\frac{1}{16\pi^2}\left(2-\log m^2+\sigma\log\frac{\sigma-1}{\sigma+1}\right)$$

factor $\frac{1}{16\pi^2}$? $\rightsquigarrow$ factored out

and d) yields:

$$B_0(p^2;m,m)-B_0(0;m,m)=2+\sigma\log\frac{\sigma-1}{\sigma+1} \rightsquigarrow B_0(0;m,m)=$$

What's in $B_0(0,m,m)$? Also $1/\epsilon$ etc.? $\rightsquigarrow$ Yes, the pole is in $B_0(p^2,\dots)$ and $B_0(0,\dots)$