Disclaimer

25.11.2017 **Advanced Quantum Field theory Exercise 7** Marvin Zanke

P3)

$$\Psi(P) \longmapsto \phi(p)\chi(q)\,[\gamma(k)]\quad,\ \Psi,\chi \text{ neutral, } \phi \text{ charged}$$

unphysical because no charge cons.? ⇝ yes and photon just a scalar particle, not gauge inv.

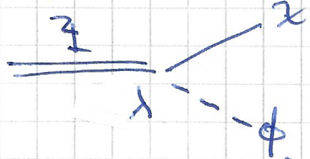
Photon scalar anyway? Or why "e.m field considered scalar"?

$$\mathcal{L} = \tfrac{1}{2}\left[(\partial\Psi)^2 - M^2\Psi^2\right] + \tfrac{1}{2}\left[(\partial\phi)^2 - m^2\phi^2\right] + \tfrac{1}{2}\left[(\partial\chi)^2 - m^2\chi\right]$$
$$+ \tfrac{1}{2}(\partial\gamma)^2 + e\gamma\phi^2 + \lambda\Psi\phi\chi$$

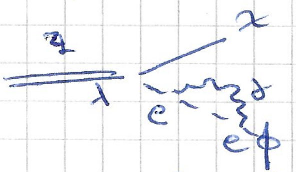
We have (A) virtual corrections and (B) Bremsstrahlung (low energy)

(a) At $\mathcal{O}(e^2)$ in the differential decay rate $d\sigma \sim |M|^2$, that means $\mathcal{O}(e)$ in M, we have to include the diagrams: ($\Psi \mathrel{\hat=}$ ═══ , $\chi \mathrel{\hat=}$ ——— , $\phi \mathrel{\hat=}$ - - - , $\gamma \mathrel{\hat=}$ 〰〰)

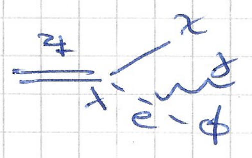
why "Bremsstrahlung only for low energies? ⇝ high energy Bremsstrahlung can be detected!



$\sim \lambda$ (M_A^0)



$\sim \lambda e^2$ (M_A^c)

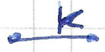


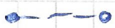
$\sim \lambda e$ (M_B)

as we add the cross sections like $\sigma(A\to B) + \sigma(A \to B\gamma) + \dots$, we will get

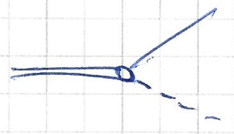
How to know that we have every diagram? Could add more λ diagrams to first one e.g.? ⇝ in-/outgoing states fixed

$$(1)\ \begin{cases} \sigma_A \sim |M_A|^2 \sim |M_A^0 + M_A^c|^2 = (M_A^0 + M_A^c)(M_A^0 + M_A^c)^* \\ \qquad = |M_A^0|^2 + M_A^0 M_A^{c*} + M_A^{0*} M_A^c \\ \qquad = |M_A^0|^2 + 2\,\mathrm{Re}(M_A^0 M_A^{c*}) \text{ at } \mathcal{O}(e^2)! \\ \sigma_B \sim |M_B|^2 \end{cases}$$

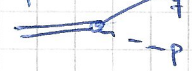
They Feynman rules yield:  $\mathrel{\hat=} \dfrac{i}{k^2 M^2}$,  $\mathrel{\hat=} \dfrac{i}{k^2 - m^2}$

 $\mathrel{\hat=} \dfrac{i}{k^2 - m^2}$,  $\mathrel{\hat=} \dfrac{i}{k^2} \left(= \dfrac{i}{k^2 - \mu^2} \right)$

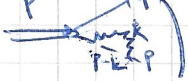
Even if ϕϕ and not ϕ*ϕ a factor 2?? ⇝ yes, need 1/2

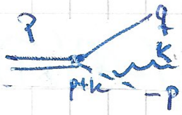
 $\mathrel{\hat=} i\lambda$,  $\mathrel{\hat=} ie$

The amplitudes are:

iM_A^0 (≙ ) $= i\lambda$

No photon mass here? ⇝ could also, but no propagator ⇝ no difference

iM_A^c (≙ ) $= (i\lambda)(ie)^2 \displaystyle\int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2} \frac{i}{(p-k)^2 - m^2} = ie^2\lambda \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} \frac{1}{(p-k)^2 - m^2}$

iM_B (≙ ) $= (i\lambda)(ie) \dfrac{i}{(p+k)^2 - m^2} = -ie\lambda \dfrac{1}{(p+k)^2 - m^2}$

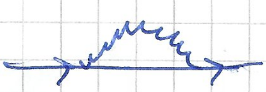
b) We will introduce a small photon mass μ

$$\rightsquigarrow \mathcal{L} \to \mathcal{L} + (-\tfrac{1}{2}\mu^2\gamma^2)$$

✓

ould also regulate w/o photon mass? ⇝ want to see the IR div.

in order to regulate the loop integral for the virtual correction $-i\Sigma(p^2)$



✓

$\Sigma(p^2)$ w/ or w/o the i from iM? ⇝ It is per convention that $-i\Sigma(p^2) =$

First, denote $[t] = [\ell] = \frac{1}{E} \rightsquigarrow [\mathcal{L}] = E^d$, as $[S = \int \mathcal{L}\, d^dx] \overset{!}{=} E^0$

$$\rightsquigarrow [\psi] = E^{d/2-1} = [\phi] = [\chi] = [\gamma]$$

$$\rightsquigarrow [e] = E^{3-d/2} = [\lambda]$$

As we want to keep the coupling constant's mass dimension in $4 \mapsto d$, we need $[e] \overset{!}{=} E^1$ and thus introduce ϱ with $[\varrho] = E^1$ in the form $\varrho^{2-d/2}$, s.t. $e \mapsto \varrho^{2-d/2} e$

$$\Rightarrow -i\Sigma(p^2) = \varrho^{4-d} e^2 \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2-\mu^2} \frac{1}{(p-k)^2-m^2}$$

ut still the whole integral diverges, so what's the use f it is IR-conv.?

$$\sim \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^4} \sim \log k \quad \text{UV-divergent}$$

$$\sim \int \frac{d^4k}{(2\pi)^4} \frac{1}{\mu^2} \frac{1}{2kp} \sim k^3 \quad \text{IR-convergent}$$

But $\Sigma'(m^2) = \frac{\partial \Sigma(p^2)}{\partial p^2}\Big|_{p^2=m^2}$ diverges for $\mu \to 0$

ow to see div. in $\Sigma'(p^2)$?

$$\rightsquigarrow -i\Sigma(p^2) = \varrho^{4-d} e^2 \int \frac{d^dk}{(2\pi)^d} \int_0^1 dx \frac{1}{\{x[(p-k)^2-m^2] + (1-x)(k^2-\mu^2)\}^2}$$

$$\Big| \; x[(p-k)^2-m^2] + (1-x)(k^2-\mu^2)$$

$$= x[p^2+k^2-2pk-m^2] + k^2 - \mu^2 - xk^2 + x\mu^2$$

$$= (k-px)^2 - p^2x^2 + x\mu^2 - \mu^2 - xm^2 + xp^2$$

$$= (k-px)^2 + x(p^2-m^2+\mu^2-p^2x) - \mu^2$$

$$= \varrho^{4-d} e^2 \int \frac{d^dk}{(2\pi)^d} \int_0^1 dx \frac{1}{\{k^2-\Delta\}^2}, \quad \Delta = \mu^2 - x(p^2-m^2+\mu^2-p^2x)$$

Wick rot. $k^0 = ik_E^0$, $k^i = k_E^i$

$$\searrow = i\varrho^{4-d} e^2 \int \frac{d^dk_E}{(2\pi)^d} \int_0^1 dx \frac{1}{(k^2+\Delta)^2} \qquad \rightsquigarrow \frac{\partial\Delta}{\partial p^2} = -x(1-x)$$

$$\rightsquigarrow \Sigma(p^2) = -\rho^{4-d} e^2 \int \frac{d^d k_E}{(2\pi)^d} \int_0^1 dx \, \frac{1}{(k^2+\Delta)^2}$$

$$\rightsquigarrow \Sigma'(p^2) = \frac{\partial \Sigma(p^2)}{\partial p^2} = -\rho^{4-d} e^2 \int \frac{d^d k_E}{(2\pi)^d} \int_0^1 dx \, \frac{(-2)\left(\frac{\partial \Delta}{\partial p^2}\right)}{(k^2+\Delta)^3}$$

$$= 2\rho^{4-d} e^2 \int \frac{d^d k_E}{(2\pi)^d} \int_0^1 dx \, \frac{x(x-1)}{(k^2+\Delta)^3}$$

$$= 2\rho^{4-d} e^2 \int_0^1 dx \, \frac{x(x-1)}{(2\pi)^d} \left(\pi^{d/2} \, \Delta^{d/2-3} \, \frac{\Gamma(3-d/2)}{\Gamma(3)} \right)$$

$$\overset{d=4}{=} e^2 \int_0^1 dx \, \frac{1}{(4\pi)^2} \Delta^{-1} \, x(x-1)$$

$$= \frac{e^2}{16\pi^2} \int_0^1 dx \, \frac{x(x-1)}{\mu^2(1-x) - p^2 x(1-x) + m^2 x}$$

$$\Rightarrow \Sigma'(m^2) = -\frac{e^2}{16\pi^2} \int_0^1 dx \, \frac{x(1-x)}{\mu^2(1-x) + m^2 x^2} \approx \frac{-\rho^{4-d} e^2}{16\pi^2} \int_0^1 dx \, \frac{x}{\mu^2 + m^2 x^2}$$

$$= -\frac{e^2}{16\pi^2} \left\{ \frac{\log(m^2 x^2 + \mu^2)}{2m^2} \right\} \Big|_0^1$$

$$= -\frac{e^2}{16\pi^2} \left\{ \frac{\log(m^2+\mu^2)}{2m^2} - \frac{\log \mu^2}{2m^2} \right]$$

$$\rightsquigarrow Z_\phi = (1 - \Sigma'(m^2))^{-1} \approx 1 + \Sigma'(m^2)$$

$$= 1 - \frac{e^2}{32\pi^2 m^2} \} \log(m^2+\mu^2) - \log \mu^2 \{$$

But also
IR-divergent
w/o derivation
here?

✓
UV finite now?
$\rightsquigarrow$ no UV-div.
in $\Sigma'(m^2)$, as
we can see after
limit $d \to 4$
(no Γ pole)
But UV-div
in $\Sigma(m^2)$!

c)

In general,

$$d\Gamma = \frac{1}{2m_A}\left(\prod_f \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\right)|M(m_A \to \{p_f\})|^2 (2\pi)^4 \delta^{(4)}(p_A - \Sigma p_f)$$

for a decaying particle A with mom. p_A and mass m_A.

Taking a look at a), we get

$$d\Gamma_A = \frac{1}{2M}\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\frac{d^3q}{(2\pi)^3}\frac{1}{2E_q}|M_A|^2(2\pi)^4\delta(P-p-q)$$

✓

Why neglect k against [illegible] integrating over k in A? ⇝ used in B

$$d\Gamma_B = \frac{1}{2M}\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\frac{d^3q}{(2\pi)^3}\frac{1}{2E_q}\frac{d^3k}{(2\pi)^3}\frac{1}{2E_k}|M_B|^2(2\pi)^4\delta(P-p-q-k)$$

$$\overset{k\approx 0}{=} \frac{1}{2M}\frac{d^3p}{(2\pi)^3 2E_p}\frac{d^3q}{(2\pi)^3 2E_q}\underbrace{\left(\frac{d^3k}{(2\pi)^3 2E_k}|M_B|^2\right)}_{\equiv d|\tilde{M}_B|^2}(2\pi)^4\delta(P-p-q)$$

✓

Why already integrated in $|\tilde{M}_B|^2$? Only differentials? ⇝ Can't detect photons → all momenta possible and thus integrated over

$$\leadsto d\Gamma^{\text{soft}}_{\text{tot}} = \frac{1}{2M}\frac{d^3p}{(2\pi)^3 2E_p}\frac{d^3q}{(2\pi)^3 2E_q}\left(|M_A|^2 + \underbrace{\int\frac{d^3k}{(2\pi)^3 2E_k}|M_B|^2}_{\equiv|\tilde{M}_B|^2}\right)(2\pi)^4\delta^4(P-p-q)$$

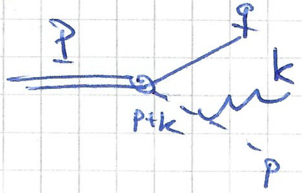
d) $|\tilde{\mathcal{M}}_B|^2 = \int \frac{d^3k}{(2\pi)^3 2E_k} |\mathcal{M}_B|^2$ with

Here, photon massless again?

$$|\mathcal{M}_B|^2 = e^2\lambda^2 \frac{1}{[(p-k)^2-m^2]^2} = e^2\lambda^2 \frac{1}{[-2pk+k^2]^2}$$

$$\simeq e^2\lambda^2 \frac{1}{4(p\cdot k)^2} = \frac{e^2\lambda^2}{4(p\cdot k)^2}$$

$p = \begin{pmatrix} m \\ 0 \end{pmatrix}$ – ϕ rest frame



„partly integrated amplitudes"?

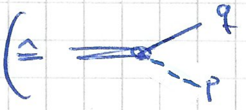
$$\rightsquigarrow |\tilde{\mathcal{M}}_B|^2 = \int \frac{d^3k}{(2\pi)^3 2E_k} \frac{e^2\lambda^2}{4(mE_k)^2}$$

res. restriction and upper cutoff

$$\overset{\curvearrowright}{=} \frac{4\pi e^2\lambda^2}{8(2\pi)^3 m^2} \int_\mu^\Lambda dk \frac{k^2}{E_k^3} = \frac{1}{16\pi^2} \frac{e^2\lambda^2}{m^2} \int_\mu^\Lambda dk \frac{k^2}{k^3}$$

$$= \frac{e^2\lambda^2}{16\pi^2 m^2} \left(\log\Lambda - \log\mu \right)$$

Taking the result of b) $Z_\phi = 1 - \frac{e^2}{32\pi^2 m^2} \{ \log(m^2+\mu^2) - \log\mu^2 \}$, we get a "new" result for $\mathcal{M}_A$:

$i\mathcal{M}_A\Big(\hat{=}$  $\Big) = i\lambda\sqrt{Z_\phi}$

$$\rightsquigarrow |\mathcal{M}_A|^2 = \lambda^2 Z_\phi = \lambda^2 \left\{ \frac{e^2}{32\pi^2 m^2} \log\mu^2 + \text{IR finite} \right\}$$

$$= \frac{e^2\lambda^2}{16\pi^2 m^2} \log\mu + \text{IR-finite}$$

In the end, we didn't need $2\mathrm{Re}(\mathcal{M}_A^0 \mathcal{M}_A^c)$ as in the lecture?

$$\rightsquigarrow |\mathcal{M}_A|^2 + |\tilde{\mathcal{M}}_B|^2 = \frac{e^2\lambda^2}{16\pi^2 m^2} \{ \log\mu - \log\mu + \log\Lambda + \text{IR finite} \}$$

$$= \frac{e^2\lambda^2}{16\pi^2 m^2} \{ \text{IR finite} \}$$

b) We will introduce a small photon mass μ WRONG!

$\rightsquigarrow \mathcal{L} \mapsto \mathcal{L} + (-\frac{1}{2}\mu^2 \gamma^2)$

in order to regulate the loop integral for the virtual correction $-i\Sigma(p^2) (= i\mathcal{M}_A^c) (\Leftrightarrow \Sigma(p^2) = -\mathcal{M}_A^c)$

could also regulate w/o photon mass?

First, denote $[t] = [\ell] = \frac{1}{E} \rightsquigarrow [\mathcal{L}] = E^d$, as $[S = \int \mathcal{L}\, d^dx] \overset{!}{=} E^0$

$\Sigma(p^2)$ w/ or w/o the i from iM?

$\rightsquigarrow [\psi] = E^{d/2-1} = [\phi] = [\chi] = [\gamma]$

$\rightsquigarrow [e] = E^{3-d/2} = [\lambda]$

As we want to keep the coupling constant's mass dimension in $4 \mapsto d$, we need $[e] \overset{!}{=} E^1$ and thus introduce ϱ with $[\varrho] = E^1$ in the form $\varrho^{2-d/2}$, s.t. $e \mapsto \varrho^{2-d/2} e$

$$\Rightarrow i\Sigma(p^2) = -4ie^2 \varrho^{4-d} \lambda \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2-\mu^2} \frac{1}{(p-k)^2-m^2}$$

But still the whole integral diverges, so what's the use if it's μ-convergent?

$$\sim \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^4} \sim \log k \text{ divergent in UV}$$

$$\sim \int \frac{d^4k}{(2\pi)^4} \frac{1}{\mu^2} \frac{1}{2kp} \sim k^3 \text{ convergent in IR}$$

But $\Sigma'(m^2) = \frac{\partial \Sigma(p^2)}{\partial p^2}\Big|_{p^2=m^2}$ diverges for $\mu \to 0$

How to see, w. in $\Sigma'(p^2)$?

$$\rightsquigarrow i\Sigma(p^2) = -4ie^2 \varrho^{4-d} \lambda \int \frac{d^dk}{(2\pi)^d} \int_0^1 dx \frac{1}{\{x[(p-k)^2-m^2] + (1-x)(k^2-\mu^2)\}^2}$$

$$\begin{aligned} &x[(p-k)^2-m^2] + (1-x)(k^2-\mu^2) \\ &= x[p^2+k^2-2pk-m^2] + k^2-\mu^2-xk^2+x\mu^2 \\ &= (k-px)^2 - p^2x^2 + x\mu^2 - \mu^2 - xm^2 + xp^2 \\ &= (k-px)^2 + x(p^2-m^2+\mu^2-p^2x) - \mu^2 \end{aligned}$$

$$= -4ie^2 \varrho^{4-d} \lambda \int \frac{d^dk}{(2\pi)^d} \int_0^1 dx \frac{1}{\{k^2 - \Delta\}^2}, \quad \Delta = \mu^2 - x(p^2-m^2+\mu^2-p^2x)$$

Wick rot. $k^0 = ik^0_E$, $k^i = k^i_E$

$$= 4e^2 \varrho^{4-d} \lambda \int \frac{d^dk_E}{(2\pi)^d} \int_0^1 dx \frac{1}{(k^2+\Delta)^2}$$

$$= \frac{4e^2 \varrho^{4-d} \lambda}{(2\pi)^d} \int_0^1 dx \left\{ \pi^{d/2} \Delta^{d/2-2} \frac{\Gamma(2-d/2)}{\Gamma(2)} \right\}$$

$$= \frac{4e^2\varrho^{4-d}\lambda}{(4\pi)^{d/2}}\,\Gamma(2-d/2)\int_0^1 dx\,\Delta^{d/2-2}$$

Now 4+2ε
dimensions

$=4-2\epsilon$

$$= \frac{4e^2\varrho^{2\epsilon}\lambda}{(4\pi)^2}(4\pi)^{\epsilon}\,\Gamma(\epsilon)\int_0^1 dx\,\Delta^{\epsilon}$$

$$= \frac{4e^2\lambda}{16\pi^2}\left(\frac{1}{\epsilon}-\gamma_E+\mathcal{O}(\epsilon)\right)\int_0^1 dx\left(\frac{4\pi\varrho^2}{\Delta}\right)^{\epsilon}, \quad x^{\epsilon}=e^{\log x\cdot\epsilon}=1+\epsilon\log x+\mathcal{O}(\epsilon^2)$$

$$= \frac{4e^2\lambda}{16\pi^2}\left(\frac{1}{\epsilon}-\gamma_E+\mathcal{O}(\epsilon)\right)\int_0^1 dx\left(1+\epsilon\log\frac{4\pi\varrho^2}{\Delta}+\mathcal{O}(\epsilon^2)\right)$$

$$\Delta=\mu^2-x(p^2-m^2-p^2x+\mu^2)=\mu^2(1-x)-p^2x(1-x)+m^2x$$

$$\approx \mu^2-p^2x+m^2x=\mu^2-x(p^2-m^2)$$

$$\int_0^1 dx\,\log\Delta \approx \int_0^1 dx\,\log(\mu^2-x(p^2-m^2)), \quad z\equiv\mu^2-x(p^2-m^2)$$

$$= \int_{\mu^2}^{\mu^2-(p^2-m^2)}\frac{dz}{m^2-p^2}\log(z)=\frac{1}{m^2-p^2}\Big[z\log z-z\Big]_{\mu^2}^{\mu^2-(p^2-m^2)} \quad dz/dx=m^2-p^2$$

$$= \frac{1}{m^2-p^2}\left\{(\mu^2-p^2+m^2)\left(\log(\mu^2-p^2+m^2)-1\right)-\mu^2(\log\mu^2-1)\right\}$$

Why not
$p^2=m^2$? Its
on shell?

$$= \frac{1}{m^2-p^2}\left\{\mu^2\log\frac{\mu^2-p^2+m^2}{\mu^2}+(m^2-p^2)\left(\log(\mu^2-p^2+m^2)-1\right)\right\}$$

$$= \frac{4e^2\lambda}{16\pi^2}\left(\frac{1}{\epsilon}-\gamma_E+\mathcal{O}(\epsilon)\right)\left(1+\epsilon\log(4\pi\varrho^2)-\epsilon\left\{\frac{\mu^2}{m^2-p^2}\log\frac{\mu^2+m^2-p^2}{\mu^2}+\log(\mu^2+m^2-p^2)-1\right\}+\mathcal{O}(\epsilon^2)\right)$$

$$= \frac{4e^2\lambda}{16\pi^2}\left\{\frac{1}{\epsilon}-\gamma_E+\log(4\pi\varrho^2)-\frac{\mu^2}{m^2-p^2}\log\frac{\mu^2+m^2-p^2}{\mu^2}-\log(\mu^2+m^2-p^2)+1+\mathcal{O}(\epsilon)\right\}$$

on shell

$$= \frac{e^2\lambda}{4\pi^2}\left\{\frac{1}{\epsilon}-\gamma_E+\log(4\pi\varrho^2)-\log\mu^2+1+\mathcal{O}(\epsilon)\right\}$$

For $\mu\to 0$, log
diverges → is th
the IR div.?
But coefficien
is also = 0?
And $\Sigma(m^2)$ shoul
be IR finite? Wh
div. now?