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06.12.2017 Advanced Quantum Field theory Exercise 8 Marvin Zanke

P.10

$$\mathcal{L} = \underbrace{-\tfrac{1}{2} H (\partial_\mu \partial^\mu + M^2) H + \bar{\psi} (i\partial\!\!\!/ - m)\psi}_{\mathcal{L}_0} \underbrace{- g \bar{\psi}\psi H}_{\mathcal{L}_{int}}$$

Why scalar field w/ of the derivative? ⇝ just integrate by parts in the action.

a)

Introducing sources, we get:

Use the $F[j] = -i\log Z[j]$ to get only fully connected diag.? ⇝ $\log Z[j]$ gives the fully! connected diagrams, while $W = \frac{Z[j]}{Z[0]}$ excludes vacuum bubbles, e.g.

In gen. fct. there are always already source terms included? ⇝ int. yes

$$\tilde{\mathcal{L}}_0 = -\tfrac{1}{2} H (\partial_\mu \partial^\mu + M^2) H + \bar{\psi}(i\partial\!\!\!/ - m)\psi + Hj + \bar{\psi}\eta + \bar{\eta}\psi$$

$H = H_{cl} + H'$, $\psi = \psi_{cl} + \psi'$

$$= -\tfrac{1}{2} (H_{cl} + H')(\partial_\mu \partial^\mu + M^2)(H_{cl} + H')$$
$$+ (\bar{\psi}_{cl} + \bar{\psi}')(i\partial\!\!\!/ - m)(\psi_{cl} + \psi')$$
$$+ (H_{cl} + H')j + (\bar{\psi}_{cl} + \bar{\psi}')\eta + \bar{\eta}(\psi_{cl} + \psi')$$

$$= \underbrace{-\tfrac{1}{2} H_{cl} (\partial_\mu \partial^\mu + M^2) H_{cl} + \bar{\psi}_{cl}(i\partial\!\!\!/ - m)\psi_{cl} + H_{cl} j + \bar{\psi}_{cl}\eta + \bar{\eta}\psi_{cl}}_{= \mathcal{L}_{cl}}$$

$$-\tfrac{1}{2} H' (\partial_\mu \partial^\mu + M^2) H' + \bar{\psi}'(i\partial\!\!\!/ - m)\psi'$$
$$+ H' j + \bar{\psi}'\eta + \bar{\eta}\psi'$$
$$-\tfrac{1}{2} H_{cl} (\partial_\mu \partial^\mu + M^2) H' - \tfrac{1}{2} H' (\partial_\mu \partial^\mu + M^2) H_{cl}$$
$$+ \bar{\psi}_{cl}(i\partial\!\!\!/ - m)\psi' + \bar{\psi}'(i\partial\!\!\!/ - m)\psi_{cl}$$

Why the i in the $H_{cl} = \ldots$ etc. solutions? ⇝ convention of the lecture (see tutorial)

The last 3 lines vanish, as we can use the (solutions for H_{cl}, ψ_{cl}, $\bar{\psi}_{cl}$ from the) classical equation of motion:

Eq. of motion from upper Lagrangian w/ $-\tfrac{1}{2}H(\ldots)H$? ⇝ $\partial_\mu H$-term? ⇝ better int. by parts and then but probably $\partial_\mu\partial^\mu H$ is a different degree of freedom!

(Prop. are the greens fcts. of the corresponding operators)

Feynman prop.

$$(\partial^2 + M^2) H_{cl} = j(x) \rightsquigarrow H_{cl} = i\int d^4y\, D_F(x-y) j(y)$$

Dirac prop.

$$(i\partial\!\!\!/ - m)\psi_{cl} = -\eta \rightsquigarrow \psi_{cl} = i\int d^4y\, S_F(x-y)\eta(y)$$

$$\bar{\psi}_{cl}(i\overleftarrow{\partial\!\!\!/} + m) = +\bar{\eta} \rightsquigarrow \bar{\psi}_{cl} = i\int d^4y\, \bar{\eta}(y) S_F(x-y)$$

and one additional integration by parts in the $\bar{\psi}_{cl}(i\partial\!\!\!/ - m)\psi'$-term $\rightarrow \bar{\psi}_{cl}(-i\overleftarrow{\partial\!\!\!/} - m)\psi'$

Thus, we have

$$\tilde{\mathcal{L}}_0 = \mathcal{L}_{cc} - \tfrac{1}{2} H' (\partial_\mu \partial^\mu + m^2) H' + \bar{\psi}' (i\not{\partial} - m) \psi'$$

For the generating functional, we notice

$$Z_{int}[j, \eta, \bar{\eta}] = \int \mathcal{D}H \int \mathcal{D}\psi \int \mathcal{D}\bar{\psi} \exp\left\{ i \int d^4x \, (\tilde{\mathcal{L}}_0 + \mathcal{L}_{int}) \right\}$$

$$= \int \mathcal{D}H \int \mathcal{D}\psi \int \mathcal{D}\bar{\psi} \exp\left\{ i \int d^4x \left(-\tfrac{1}{2} H (\partial_\mu \partial^\mu + m^2) H + \bar{\psi}(i\not{\partial} - m)\psi + Hj + \bar{\psi}\eta + \bar{\eta}\psi - g \bar{\psi}\psi H \right) \right\}$$

$= \langle 0 | 0 \rangle^{j, \eta, \bar{\eta}}$
Because actually
had $\langle 0 | 0 \rangle^{j\eta\bar{\eta}} = e^{\int d \ldots}$
$\langle 0 | 0 \rangle$, not w/ $Z[\ldots$
$\Rightarrow$ interpret
as vacuum to
vacuum transition
amplitude (sam…

$$= \int \mathcal{D}H \int \mathcal{D}\psi \int \mathcal{D}\bar{\psi} \exp\left\{ -ig \int d^4z \, \bar{\psi}\psi H \right\} \times \exp\left\{ i \int d^4x \left(-\tfrac{1}{2} H (\partial_\mu \partial^\mu + m^2) H + \bar{\psi}(i\not{\partial} - m)\psi + Hj + \bar{\psi}\eta + \bar{\eta}\psi \right) \right\}$$

$$= \int \mathcal{D}H \int \mathcal{D}\psi \int \mathcal{D}\bar{\psi} \exp\left\{ -ig \int d^4z \left(\frac{1}{i} \frac{\delta}{\delta \eta(z)} \, \frac{1}{i} \frac{\delta}{\delta \bar{\eta}(z)} \, \frac{1}{i} \frac{\delta}{\delta j(z)} \right) \right\} \times \exp\left\{ i \int d^4x \, \tilde{\mathcal{L}}_0 \right\}$$

$$= \int \mathcal{D}H \int \mathcal{D}\psi \int \mathcal{D}\bar{\psi} \exp\left\{ g \int d^4z \left(\frac{\delta}{\delta \eta(z)} \frac{\delta}{\delta \bar{\eta}(z)} \frac{\delta}{\delta j(z)} \right) \right\} \times \exp\left\{ i \int d^4x \, \tilde{\mathcal{L}}_0 \right\}$$

$$= \exp\left\{ g \int d^4z \left(\frac{\delta}{\delta \eta(z)} \frac{\delta}{\delta \bar{\eta}(z)} \frac{\delta}{\delta j(z)} \right) \right\} \underbrace{\left\{ \int \mathcal{D}H \int \mathcal{D}\psi \int \mathcal{D}\bar{\psi} \exp\left\{ i \int d^4x \, \tilde{\mathcal{L}}_0 \right\} \right\}}_{Z[j, \eta, \bar{\eta}] \text{ (no int. T)}}$$

Using the above result, we find:

$$Z[j, \eta, \bar{\eta}] = \int \mathcal{D}H' \int \mathcal{D}\psi' \int \mathcal{D}\bar{\psi}' \exp\left\{ i \int d^4x \left(\mathcal{L}_{cc} - \tfrac{1}{2} H' (\partial_\mu \partial^\mu + m^2) H' + \bar{\psi}'(i\not{\partial} - m)\psi' \right) \right\}$$

$$= \exp\left(i \int d^4x \, \mathcal{L}_{cc} \right) \int \mathcal{D}H' \int \mathcal{D}\psi' \int \mathcal{D}\bar{\psi}' \exp\left\{ i \int d^4x \left(-\tfrac{1}{2} H' (\partial_\mu \partial^\mu + m^2) H' + \bar{\psi}'(i\not{\partial} - m)\psi' \right) \right\}$$

$$= \exp\left(i \int d^4x \, \mathcal{L}_{cc} \right) Z[j=0, \eta=0, \bar{\eta}=0]$$

Taking a closer look at $iS_{cl} \equiv i\int d^4x \mathcal{L}_{cl}$, one finds:

$$iS_{cl} = i\int d^4x \Big\{ -\tfrac{1}{2} H_{cl}(\partial_\mu\partial^\mu + M^2) H_{cl} + \bar{\psi}_{cl}(i\not\partial - m)\psi_{cl} + H_{cl}\, j + \bar{\psi}_{cl}\,\eta + \bar{\eta}\,\psi_{cl} \Big\}$$

Sol. of the
cl. eq. of motion

$$= i\int d^4x \Big\{ -\tfrac{1}{2} H_{cl}^{(x)} j(x) - \bar{\psi}_{cl}^{(x)}\eta(x) + H_{cl}^{(x)} j(x) + \bar{\psi}_{cl}^{(x)}\eta(x) + \bar{\eta}(x)\psi_{cl}^{(x)} \Big\}$$

$$= i\int d^4x \Big\{ \tfrac{1}{2} H_{cl}^{(x)} j(x) + \bar{\eta}(x)\psi_{cl}^{(x)} \Big\}$$

$$= -\int d^4x\, d^4y \Big\{ \tfrac{1}{2} j(y) D_F(x-y) j(x) + \bar{\eta}(x) S_F(x-y)\eta(y) \Big\}$$

All in all, we thus get:

$$Z_{int}[j,\eta,\bar{\eta}] = \exp\Big\{ g\int d^4z \Big(\frac{\delta}{\delta\eta(z)}\frac{\delta}{\delta\bar{\eta}(z)}\frac{\delta}{\delta j(z)}\Big)\Big\} \times Z[j=0,\eta=0,\bar{\eta}=0]$$
$$\times \exp\Big\{ -\int d^4x\, d^4y \Big(\tfrac{1}{2} j(y) D_F(x-y) j(x) + \bar{\eta}(x) S_F(x-y)\eta(y)\Big)\Big\}$$

We further define the generating functional:

Is Z or W "the" gen. fct.? ✓
depends on what you call it.

$$W_{int}[j,\eta,\bar{\eta}] = \frac{Z_{int}[j,\eta,\bar{\eta}]}{Z_{int}[j=0,\eta=0,\bar{\eta}=0]}$$

$$= \frac{\exp\Big\{ g\int d^4z \Big(\frac{\delta}{\delta\eta(z)}\frac{\delta}{\delta\bar{\eta}(z)}\frac{\delta}{\delta j(z)}\Big)\Big\} \exp\Big\{ -\int d^4x\, d^4y \Big(\tfrac{1}{2} j(y) D_F(x-y) j(x) + \bar{\eta}(x) S_F(x-y)\eta(y)\Big)\Big\}}{\exp(\ldots)\quad \exp(\ldots)\Big|_{j=0,\eta=0,\bar{\eta}=0}}$$

b) If we do not couple the H-field to an external source, we get:

$$\tilde{\mathcal{L}}_0 = -\tfrac{1}{2} H(\partial_\mu \partial^\mu + M^2)H + \bar{\psi}(i\not{\partial} - m)\psi + \bar{\psi}\eta + \bar{\eta}\psi$$
$$\tilde{\mathcal{L}}_0 + \mathcal{L}_{int} = \mathcal{L} + \bar{\psi}\eta + \bar{\eta}\psi = \tilde{\mathcal{L}}$$

Defining $\hat{D}_H = \partial^2 + M^2$, $\hat{D}_\psi = i\not{\partial} - m$, $\xi = g\bar{\psi}\psi$, we get

$$\tilde{Z}[\eta, \bar{\eta}] = \int \mathcal{D}H\, \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\, \exp\left\{ i \int d^4x \left\{ -\tfrac{1}{2} H \hat{D}_H H + \bar{\psi}\hat{D}_\psi \psi - \xi H + \bar{\psi}\eta + \bar{\eta}\psi \right\}\right\}$$

$$= \int \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\, \exp\left\{ i \int d^4x \left(\bar{\psi} \hat{D}_\psi \psi + \bar{\psi}\eta + \bar{\eta}\psi\right)\right\} \times \int \mathcal{D}H \exp\Big\{ i \underbrace{\int d^4x \left(-\tfrac{1}{2} H \hat{D}_H H - \xi H\right)}_{=T} \Big\}$$

Split the 3 integrals like this?

$T = \int d^4x \left(-\tfrac{1}{2} H \hat{D}_H H - \xi H\right)$, using the shorthand notation to integrate over indices appearing twice, and noticing

$$\int d^4x \left(-\tfrac{1}{2} H D_H H\right) = \int d^4x\, d^4y \Big(-\tfrac{1}{2} H_x \underbrace{\left[\delta(x-y)(\Box^2 + M^2)\right]}_{(D_H)_{xy}} H_y\Big)$$

$$\rightsquigarrow T = -\tfrac{1}{2} H_x (D_H)_{xy} H_y - \xi_x H_x$$

$$\overset{H_x \mapsto H_x' - (D_H^{-1})_{xz}\xi_z}{=} -\tfrac{1}{2}\left(H_x' - (D_H^{-1})_{xz}\xi_z\right)(D_H)_{xy}\left(H_y' - (D_H^{-1})_{yw}\xi_w\right) - \xi_x\left(H_x' - (D_H^{-1})_{xu}\xi_u\right)$$

$$= -\tfrac{1}{2} H_x' (D_H)_{xy} H_y' + \tfrac{1}{2} H_x' \overbrace{(D_H)_{xy}(D_H^{-1})_{yw}}^{=\mathbb{1}_{xw}} \xi_w + \tfrac{1}{2} \underbrace{(D_H^{-1})_{xz}\xi_z (D_H)_{xy}}_{\mathbb{1}_{zy}} H_y' - \tfrac{1}{2}(D_H^{-1})_{xz}\xi_z \underbrace{(D_H)_{xy}(D_H^{-1})_{yw}}_{=\mathbb{1}_{xw}}\xi_w - \xi_x H_x' + \xi_x (D_H^{-1})_{xu}\xi_u$$

$$= -\tfrac{1}{2} H_x' (D_H)_{xy} H_y' + \tfrac{1}{2} H_x' \xi_x + \tfrac{1}{2} \xi_y H_y' - \tfrac{1}{2} (D_H^{-1})_{xz} \xi_z \xi_x$$

$$- \xi_x H_x' + \xi_x (D_H^{-1})_{xu} \xi_u$$

rename vars $= -\tfrac{1}{2} H_x' (D_H)_{xy} H_y' - \tfrac{1}{2} \xi_x (D_H^{-1})_{xz} \xi_z + \xi_x (D_H^{-1})_{xu} \xi_u$

$$= -\tfrac{1}{2} H_x' (D_H)_{xy} H_y' + \tfrac{1}{2} \xi_x (D_H^{-1})_{xz} \xi_z$$

c) $\mathcal{L}_{int} = \frac{ig^2}{2}\int d^4y\; \bar{\psi}(x)\psi(x)\, D_F(x-y)\, \bar{\psi}(y)\psi(y)$

Greens fct. of D_H ($\hat{=} D_H^{-1}$)

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-M^2} e^{-ip(x-y)}$$

Why cutoff if $p^2 \ll M^2$? Or how to set $p^2 \ll M^2$ if integrating over it?

$$= i\int \frac{d^4p}{(2\pi)^4} \frac{1/M^2}{p^2/M^2 - 1} e^{-ip(x-y)}$$

$$= i\int \frac{d^4p}{(2\pi)^4} \frac{1}{M^2}\left(-1 - \mathcal{O}\left(\frac{p^2}{M^2}\right)\right) e^{-ip(x-y)}$$

$$= -\frac{i}{M^2}\int_{\Lambda^2} \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \left(1 + \mathcal{O}\left(\frac{p^2}{M^2}\right)\right)$$

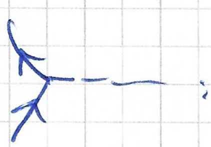
$$= -\frac{i}{M^2}\delta^{(4)}(x-y) + \mathcal{O}\left(\frac{\Lambda^2}{M^2}\right)$$

$$\rightsquigarrow \mathcal{L}_{int}^{eff} \approx \frac{g^2}{2M^2}\left(\bar{\psi}(x)\psi(x)\right)\left(\bar{\psi}(x)\psi(x)\right)$$

✓

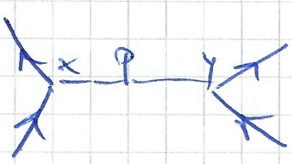
what does it physically mean that it's local? → non-local as we have a propagator connecting 2 spacetime points → local: only interaction at same spacetime. (usually the prop. is also an interaction between 2 spacetime points, but its not in an integral d^4y and appearing in the vertex)

d) i)



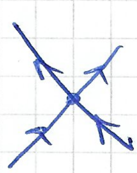
: $-ig$

ii)



: $-\frac{g^2}{2} D_F(x-y)$ ($\times$ sym. factor ?)

iii)



: $\frac{ig^2}{2M^2}$ ($\times$ sym. factor?)