Disclaimer

The solution at hand was written in the course of the respective class at the University of Bonn. If not stated differently on top of the first page or the following website, the solution was prepared and handed in solely by me, Marvin Zanke. Anything in a different color than the ball pen blue is usually a correction that I or a tutor made. For more information and all my material, check:
https://www.physics-and-stuff.com/

I raise no claim to correctness and completeness of the given solutions! This equally applies to the corrections mentioned above.

22.10.2016 Advanced Quantum Theory Exercise 1 Marvin Zanke

Great

H1) d)

Given $v(\vec{q}) = \sqrt{\frac{\Lambda^3}{m}} \frac{1}{\vec{q}^2+\Lambda^2}$ with $v(\vec{q}) = v(|\vec{q}|)$ (1)

we get with $B(z) \overset{(b)}{=} \int d^3q \frac{1}{(2\pi\hbar)^3} \frac{|v(q)|^2}{z - E_q}$

$$\Rightarrow \lim_{\epsilon\to 0^+} B(E_p + i\epsilon) = \lim_{\epsilon\to 0^+} \int \frac{d^3q}{(2\pi\hbar)^3} \frac{\Lambda^3}{m} \frac{1}{(\vec{q}^2+\Lambda^2)^2} \frac{1}{E_p + i\epsilon - E_q}$$

What is a seperable potential and a dipole form?

Why $\epsilon \to 0^+$ and not $\epsilon \to 0$?

$$= (-1) \lim_{\epsilon\to 0^+} \left(-\frac{\Lambda^3}{2\pi^2\hbar^3}\right) \frac{1}{4\pi} \frac{1}{m} \int d^3q \frac{1}{(\vec{q}^2+\Lambda^2)^2} \frac{1}{(E_p - E_q) + i\epsilon}$$

$$= \left(-\frac{\Lambda^3}{2\pi^2\hbar^3}\right)(-1) \lim_{\epsilon\to 0^+} \frac{1}{4\pi} \frac{1}{m} \int_0^\infty dq \int_0^{2\pi} d\varphi \int_0^\pi d\vartheta\, q^2 \sin\vartheta \frac{1}{(\vec{q}^2+\Lambda^2)^2} \frac{1}{(E_p - E_q)+i\epsilon}$$

no ϑ, φ dependence but $\sin\vartheta$ of fct. det. $\Rightarrow 4\pi \hat{=}$ solid angle

$$= \left(-\frac{\Lambda^3}{2\pi^2\hbar^3}\right)(-1) \lim_{\epsilon\to 0^+} \frac{1}{m} \int_0^\infty dq\, q^2 \frac{1}{(\vec{q}^2+\Lambda^2)^2} \frac{1}{\frac{1}{2m}(\vec{p}^2-\vec{q}^2)+i\epsilon}$$

$E_p = \frac{p^2}{2m}$

$$= \left(-\frac{\Lambda^3}{2\pi^2\hbar^3}\right)(-1) \lim_{\epsilon\to 0^+} \frac{2m}{m} \int_0^\infty dq\, q^2 \frac{1}{(q^2+\Lambda^2)^2} \frac{1}{p^2 - q^2 + 2mi\epsilon}$$

$$= \left(-\frac{\Lambda^3}{2\pi^2\hbar^3}\right) \lim_{\epsilon\to 0^+} 2 \cdot \int_0^\infty dq\, h(q)$$

$h(-q) = h(q)$

$$= \left(-\frac{\Lambda^3}{2\pi^2\hbar^3}\right) \lim_{\epsilon\to 0^+} \int_{-\infty}^{\infty} dq\, h(q)$$

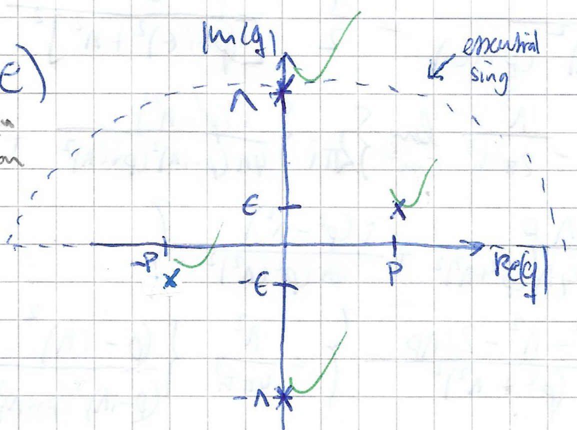
$$h(q) = \frac{q^2}{(q-i\Lambda)^2(q+i\Lambda)^2[q-(p+i\epsilon')][q+(p+i\epsilon')]}$$

$$= \frac{q^2}{(q^2+\Lambda^2)^2[q^2-(p+i\epsilon')^2]} = \frac{q^2}{(q^2+\Lambda^2)^2[q^2-p^2+\epsilon'^2-2ip\epsilon']}$$

$\epsilon'^2 \to 0$ in $\lim_{\epsilon\to 0}$

$$= \frac{-q^2}{(q^2+\Lambda^2)^2[p^2-q^2+2ip\epsilon']} \quad \text{with } \epsilon' = \frac{m}{p}\cdot\epsilon$$

e)



Better argumentation why the expandation doesn't change the value of the integral?

Integration has to be expanded to a closed curve in order to be able to use the residue theorem.

As $t \mapsto Re^{it}$ parametrizes the arc, one can easily see that for $R \to \infty$, all contributions with $\mathrm{Im}(q) \overset{?}{\underset{(>)}{=}} 0$ result in 0, because $h(q) \xrightarrow{t\to\infty} 0$

$$\lim_{\epsilon\to 0^+} B(E_p+i\epsilon) = -\frac{\Lambda^3}{2\pi^2\hbar^3}\lim_{\epsilon\to 0^+}\int_{-\infty}^{\infty} dq\, h(q) \overset{\substack{\text{expand}\\\text{to}\\\text{closed}\\\text{curve}\\\text{as sketched}}}{=} -\frac{\Lambda^3}{2\pi^2\hbar^3}\lim_{\epsilon\to 0^+}\underbrace{\int_{C^+} dq\, h(q)}_{2\pi i \sum_{k=1}^{n} I(C^+,a_k)\,\mathrm{Res}(h,a_k)} \qquad (2)$$

To calculate the residuums, we use the given fact

$$\mathrm{Res}(h,i\Lambda) = \frac{1}{1}\frac{d}{dq}\left[(q-i\Lambda)^2 h(q)\right]\Big|_{q=i\Lambda} = \frac{d}{dq}\frac{q^2}{(q+i\Lambda)[q-(p+i\epsilon)][q+(p+i\epsilon)]}\Big|_{q=i\Lambda}$$

$$= \frac{2q\left\{(q+i\Lambda)^2\left[q^2-(p+i\epsilon)^2\right]\right\} - q^2\left\{2(q+i\Lambda)\left[q^2-(p+i\epsilon)^2\right]+2q(q+i\Lambda)^2\right\}}{(q+i\Lambda)^4\left[q^2-(p+i\epsilon)^2\right]^2}\Bigg|_{q=i\Lambda}$$

$$= \frac{2i\Lambda\left\{4\Lambda^2\left[\Lambda^2+(p+i\epsilon)^2\right]\right\} + \Lambda^2\left\{4i\Lambda\left[-\Lambda^2-(p+i\epsilon)^2\right]-8i\Lambda^3\right\}}{16\Lambda^4\left[\Lambda^2+(p+i\epsilon)^2\right]^2}$$

$$= \frac{8i\Lambda^3\left[\Lambda^2+(p+i\epsilon)^2\right]-4i\Lambda^3\left[\Lambda^2+(p+i\epsilon)^2\right]-8i\Lambda^5}{16\Lambda^4\left[\Lambda^2+(p+i\epsilon)^2\right]^2}$$

$$= \frac{4i\Lambda^3\left[\Lambda^2+(p+i\epsilon)^2\right]-8i\Lambda^5}{16\Lambda^4\left[\Lambda^2+(p+i\epsilon)^2\right]^2} = \frac{i\left[\Lambda^2+p^2+2ip\epsilon-\epsilon^2\right]-2i\Lambda^2}{4\Lambda\left[\Lambda^2+p^2+2i\epsilon p-\epsilon^2\right]^2}$$

$$= \frac{i\left[p^2+2ip\epsilon-\epsilon^2-\Lambda^2\right]}{4\Lambda\left[p+i\epsilon-i\Lambda\right]^2\left[p+i\epsilon+i\Lambda\right]^2}$$

$$\mathrm{Res}(h,p+i\epsilon) = \frac{1}{0!}\frac{d}{dq}\left[(q-(p+i\epsilon))h(q)\right]\Big|_{q=p+i\epsilon}$$

$$= \frac{q^2}{(q-i\Lambda)^2(q+i\Lambda)^2\left[q+(p+i\epsilon)\right]}\Bigg|_{q=p+i\epsilon}$$

$$= \frac{(p+i\epsilon)^2}{2(p+i\epsilon-i\Lambda)^2(p+i\epsilon+i\Lambda)^2(p+i\epsilon)} = \frac{1}{2}\frac{p+i\epsilon}{\left[(p+i\epsilon)^2+\Lambda^2\right]^2}$$

So that (2) yields:

$$\lim_{\epsilon\to 0^+} B(E_p+i\epsilon) = -\frac{\Lambda^3}{2\pi^2\hbar^3}\lim_{\epsilon\to 0^+} 2\pi i\left\{\frac{i(p^2-\Lambda^2)}{4\Lambda(p-i\Lambda)^2(p+i\Lambda)^2}+\frac{p}{2(p^2+\Lambda^2)^2}\right\}$$

$$= -\frac{i\Lambda^3}{\pi\hbar^3}\left\{\frac{2\Lambda p}{4\Lambda(p^2+\Lambda^2)^2}+\frac{i(p^2-\Lambda^2)}{4\Lambda(p^2+\Lambda^2)^2}\right\}$$

$$= \frac{\Lambda^3}{4\Lambda\pi\hbar^3}\left\{\frac{p^2-\Lambda^2-2i\Lambda p}{(p^2+\Lambda^2)^2}\right\} = \frac{\Lambda^2}{4\pi\hbar^3}\left\{\frac{(p-i\Lambda)^2}{(p-i\Lambda)^2(p+i\Lambda)^2}\right\}$$

$$= \frac{1}{4\pi\hbar^3}\frac{\Lambda^2}{(p+i\Lambda)^2}$$

$\square$

H2)

1) $I_r^m(f) = \int_{\partial_r^m} dz\, f(z) = \int_0^{2\pi} dt\, (\partial \gamma_r^m)(t)\, f(\gamma_r^m(t))\,,\ m \in \mathbb{Z}\setminus\{0\}$

$\gamma_r^m(t) = re^{imt}, r>0;\quad f(z) = z^n, n \in \mathbb{Z}$

$\rightsquigarrow I_r^m(f) = \int_0^{2\pi} dt\ r(im)e^{imt}\,(re^{imt})^n = r^{n+1}\, im \int_0^{2\pi} dt\, e^{(n+1)imt}$

always = 1: $e^{2\pi i m(n+1)}$ for $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$?

$= im\, r^{n+1} \frac{1}{im(n+1)} \left[e^{(n+1)imt} \right]\Big|_0^{2\pi}$

$= \frac{r^{n+1}}{n+1} \left\{ (e^{2\pi i n(n+1)}) - e^0 \right\} = 0$

NOT TO BE CORRECTED: (From Tutorial): $2\pi i m\, \delta_{n,-1}$

No dependence on r and m (of course, because f(z) is holomorphic and the integral over a closed curve of a holomorphic function vanishes).

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m is the number of turns the path $\gamma_r^m(t)$ makes around zero during $0 \leq t \leq 2\pi$.
(Or the speed of the path?)

Is m the speed or just the number of turns?

Looking for F, such that $(\partial_z F)(z) = f(z)$ along complete path
Claim that: $F = \frac{1}{n+1} z^{n+1}$ ~~satisfies this condition,~~
~~except for $n = -1$~~

Also negative n?

Tutorial: for $n = -1$ exists $F(z) = \ln(z)$

ii) $f(z) = \exp(z)$

$\rightsquigarrow I_r^m(f) = \int_0^{2\pi} dt\ imre^{imt}\, e^{re^{imt}} = \int_0^{2\pi} dt\ imre^{imt} \sum_{n=0}^{\infty} \frac{(re^{imt})^n}{n!}$

$= im \int_0^{2\pi} dt \underbrace{\sum_{n=0}^{\infty} \frac{(re^{imt})^{n+1}}{n!}}_{\geq 0 \text{ as } r>0} \underset{\text{Tonelli}}{=} im \sum_{n=0}^{\infty} \int_0^{2\pi} dt \frac{(re^{imt})^{n+1}}{n!}$

Do you need $f_n \geq 0$ to interchange integral and sum?

$= im \sum_{n=0}^{\infty} \frac{1}{n!} r^{n+1} \frac{1}{im} \left\{ e^{imt} \right\}\Big|_0^{2\pi} = \sum_{n=0}^{\infty} \frac{1}{n!} r^{n+1} \{1-1\} =0$

iii) $f(z) = \sum_{n=-3}^{\infty} (-1)^n \frac{3n^2 \sin\left(\frac{\pi}{2} \frac{1+2n}{3}\right)}{2^n \pi (n+4)!} z^n$

$$\rightsquigarrow I_r^m(f) = \int_0^{2\pi} dt\, imre^{imt} \sum_{n=-3}^{\infty} (-1)^n \frac{3n^2 \sin\left(\frac{\pi}{2} \frac{1+2n}{3}\right)}{2^n \pi (n+4)!} \left(re^{imt}\right)^n$$

$$= \frac{im}{\pi} \int_0^{2\pi} dt \underbrace{\sum_{n=-3}^{\infty} (-1)^n \frac{3n^2 \sin\left(\frac{\pi}{2} \frac{1+2n}{3}\right)}{2^n (n+4)!} \left(re^{imt}\right)^{n+1}}_{=: f_n}$$

necessary?

and $K := \sum |f_n| < \infty$ because $\frac{n^2}{2^n} \to 0$ and $\frac{e^{imt(n+1)}}{(n+4)!} \to 0$ as $n \to \infty$

and $|\sin(x)| \leq 1 \ \forall x$ and $\int K < \infty$ because t dependence only in $e^{imt(n+1)}$

Fubini

$$= \frac{im}{\pi} \sum_{n=-3}^{\infty} \int_0^{2\pi} dt (-1)^n \frac{3n^2 \sin\left(\frac{\pi}{2} \frac{1+2n}{3}\right)}{2^n (n+4)!} \left(re^{imt}\right)^{n+1}$$

$$= \frac{im}{\pi} \sum_{n=-3}^{\infty} (-1)^n \frac{3n^2 \sin\left(\frac{\pi}{2} \frac{1+2n}{3}\right)}{2^n (n+4)!} r^{n+1} \underbrace{\int_0^{2\pi} dt\, e^{imt(n+1)}}_{= 0, \text{ as already proven}}$$

Not 0 for $n = -1$!

$= 0$ ×

Tutorial:

$$I_r^m(f) = 2\pi i m (-1)^{-1} \frac{3(-1)^2 \sin\left(\frac{\pi}{2} \frac{1+2(-1)}{3}\right)}{2^{-1} \pi (-1+4)!}$$

$= im$

$f(z) = \sum_{m=-\infty}^{\infty} a_m (z-z_0)^m \rightsquigarrow a_{-1}$

$\oint f(z) = 2\pi i a_{-1}$