12.01.2017 Advanced Quantum theory Exercise 10

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A13) Dirac equation : $(i\gamma^\mu \partial_\mu - m)\psi(x) = 0$

pos. energy solution : $\psi_r^+(x) = e^{-ip_\mu x^\mu} u_r(p)$

Depends on p as well?

$\rightsquigarrow (i\gamma^\mu\partial_\mu - m)\psi_r^{(+)}(x) = 0 \Leftrightarrow i\gamma^\mu(-ip_\mu)e^{-ip_\mu x^\mu}u_r(p) - me^{-ip_\mu x^\mu}u_r(p) = 0$

$\Leftrightarrow (\not{p} - m)u_r(p) = 0$ (1)

$\Leftrightarrow u_r^+(p)(\gamma^\mu p_\mu - m)^+ = 0 \Leftrightarrow u_r^+(p)(p_\mu^+(\gamma^\mu)^+ - m) = 0$

$\underset{(\gamma^\mu)^+ = \gamma^0\gamma^\mu\gamma^0}{\overset{p^+ = p}{\Leftrightarrow}} u_r^+(p)(p_\mu\gamma^0\gamma^\mu\gamma^0 - m) = 0 \Leftrightarrow \bar{u}_r(p)(p_\mu\gamma^\mu - m) = 0$

$\Leftrightarrow \bar{u}_r(p)(\not{p} - m) = 0$ (2)

Now with $\sigma^{\mu\nu} := \frac{i}{2}[\gamma^\mu, \gamma^\nu]$, we look at RHS:

$\frac{1}{2m}\bar{u}_r(p')[(p'+p)^\mu + i\sigma^{\mu\nu}(p'-p)_\nu]u_s(p)$

$= \frac{1}{2m}\bar{u}_r(p')(p'^\mu + p^\mu)u_s(p) - \frac{1}{4m}\bar{u}_r(p')[\gamma^\mu, \gamma^\nu](p'_\nu - p_\nu)u_s(p)$

$= \frac{1}{2m}\bar{u}_r(p')p'^\mu u_s(p) + \frac{1}{2m}\bar{u}_r(p')p^\mu u_s(p) - \frac{1}{4m}\bar{u}_r(p')\gamma^\mu\gamma^\nu p'_\nu u_s(p)$
$+ \frac{1}{4m}\bar{u}_r(p')\gamma^\mu\gamma^\nu p_\nu u_s(p) + \frac{1}{4m}\bar{u}_r(p')\gamma^\nu\gamma^\mu p'_\nu u_s(p) - \frac{1}{4m}\bar{u}_r(p')\gamma^\nu\gamma^\mu p_\nu u_s(p)$

$= \frac{1}{2m}\bar{u}_r(p')p'^\mu u_s(p) + \frac{1}{2m}\bar{u}_r(p')p^\mu u_s(p) - \frac{1}{4m}\bar{u}_r(p')\{2g^{\mu\nu} - \gamma^\nu\gamma^\mu\}p'_\nu u_s(p)$
$+ \frac{1}{4m}\bar{u}_r(p')\gamma^\mu\gamma^\nu p_\nu u_s(p) + \frac{1}{4m}\bar{u}_r(p')\gamma^\nu\gamma^\mu p'_\nu u_s(p) - \frac{1}{4m}\bar{u}_r(p')\gamma^\nu\gamma^\mu p_\nu u_s(p)$

$= \frac{1}{2m}\bar{u}_r(p')p^\mu u_s(p) + \frac{1}{4m}\bar{u}_r(p')\gamma^\mu\gamma^\nu p_\nu u_s(p) + \frac{1}{2m}\bar{u}_r(p')\gamma^\nu\gamma^\mu p'_\nu u_s(p)$
$- \frac{1}{4m}\bar{u}_r(p')\gamma^\nu\gamma^\mu p_\nu u_s(p)$

$= \frac{1}{2m}\bar{u}_r(p')p^\mu u_s(p) + \frac{1}{4m}\bar{u}_r(p')\gamma^\mu\gamma^\nu p_\nu u_s(p) + \frac{1}{2m}\bar{u}_r(p')\gamma^\nu\gamma^\mu p'_\nu u_s(p)$
$- \frac{1}{4m}\bar{u}_r(p')\{2g^{\mu\nu} - \gamma^\mu\gamma^\nu\}p_\nu u_s(p)$

$= \frac{1}{2m}\bar{u}_r(p')\gamma^\mu\gamma^\nu p_\nu u_s(p) + \frac{1}{2m}\bar{u}_r(p')\gamma^\nu\gamma^\mu p'_\nu u_s(p)$

Just pull p'_ν in front? $\gamma^\nu\gamma^\mu p'_\nu = p'_\nu\gamma^\nu\gamma^\mu$?

$= \frac{1}{2m}[\bar{u}_r(p')\gamma^\mu \underbrace{\not{p}u_s(p)}_{(1)} + \underbrace{\bar{u}_r(p')\not{p}'}_{(2)}\gamma^\mu u_s(p)]$

$= \frac{1}{2m}[\bar{u}_r(p')\gamma^\mu m u_s(p) + \bar{u}_r(p')m\gamma^\mu u_s(p)]$

$= \frac{1}{2}[\bar{u}_r(p')\gamma^\mu u_s(p) + \bar{u}_r(p')\gamma^\mu u_s(p)]$

$= \bar{u}_r(p')\gamma^\mu u_s(p)$ ☒

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H14) As we know from H13): (1) $(\not p - m) u_r(p) = 0$, $\bar{u}_r(p)(\not p - m) = 0$ pos. energy solutions

analogue (2) $(\not p + m) v_r(p) = 0$, $\bar{v}(p)(\not p + m) = 0$ neg. energy solutions

Now define: $\Lambda_\pm := \frac{m \pm \not p}{2m}$

Normalization: Besides that, we showed for the explicit expressions of the spinors (lecture):

Same N for both?

$$u_r(p) = N(\not p + m) u_r(m, \vec{0}) = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} \chi_r \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \chi_r \end{pmatrix}$$

$$v_r(p) = N(\not p - m) v_r(m, \vec{0}) = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \chi_r \\ \chi_r \end{pmatrix}$$

with $\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Why not use the explicit expressions we derived?

a) $\Lambda_+ u_r = \frac{m + \not p}{2m} u_r \overset{(1)}{=} \frac{m+m}{2m} u_r = u_r$ ✓

$\Lambda_+ v_r = \frac{m + \not p}{2m} v_r \overset{(2)}{=} \frac{m-m}{2m} v_r = 0$ ✓

$\Lambda_- u_r = \frac{m - \not p}{2m} u_r \overset{(1)}{=} \frac{m-m}{2m} u_r = 0$ ✓

$\Lambda_- v_r = \frac{m - \not p}{2m} v_r \overset{(2)}{=} \frac{m-(-m)}{2m} v_r = v_r$ ✓ 4/4

b) $\Lambda_\pm^2 = \frac{m \pm \not p}{2m} \cdot \frac{m \pm \not p}{2m} = \frac{m^2 + \not p^2 \pm 2m\not p}{4m^2} = \frac{2m^2 \pm 2m\not p}{4m^2}$

$= \frac{m \pm \not p}{2m} = \Lambda_\pm$ ✓

where we used that $\not p^2 = \gamma^\mu p_\mu \gamma^\nu p_\nu = \frac{1}{2}\gamma^\mu\gamma^\nu p_\mu p_\nu + \frac{1}{2}\gamma^\nu\gamma^\mu p_\nu p_\mu$

$= \frac{1}{2} p_\mu p_\nu \{\gamma^\mu, \gamma^\nu\} = \frac{1}{2} p_\mu p_\nu (2 g^{\mu\nu} \mathbb{1})$

$= p^\mu p_\mu \mathbb{1} = (E^2 - \vec{p}^2)\mathbb{1} = m^2 \mathbb{1}$ ✓

Why does $[\not p, \not p]$ commute?

$\Lambda_+ \Lambda_- = \frac{m + \not p}{2m} \cdot \frac{m - \not p}{2m} = \frac{m - \not p}{2m} \cdot \frac{m + \not p}{2m} = \Lambda_- \Lambda_+$

$= \frac{m^2 - \not p^2}{4m^2} = \frac{m^2 - m^2}{4m^2} = 0$ ✓

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c) We will use $u_r(p) = N\begin{pmatrix} \chi_r \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_r \end{pmatrix}$, $v_r(p) = \begin{pmatrix} N\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_r \\ \chi_r \end{pmatrix}$

Which of the explicit expressions for u_r, v_r is meant?

with $N = \sqrt{\frac{E+m}{2m}}$, $\chi_1 = \begin{pmatrix}1\\0\end{pmatrix}$, $\chi_2 = \begin{pmatrix}0\\1\end{pmatrix}$

We first find: $\Lambda_\pm(p) = \frac{m \pm p_\mu\gamma^\mu}{2m} = \frac{1}{2m}\left\{\begin{pmatrix} m\mathbb{1} & 0 \\ 0 & m\mathbb{1}\end{pmatrix} \pm \begin{pmatrix} p_0\mathbb{1} & 0 \\ 0 & -p_0\mathbb{1}\end{pmatrix}\right.$

$\left.\pm \begin{pmatrix} 0 & p_1\sigma^1 \\ -p_1\sigma^1 & 0\end{pmatrix} \pm \begin{pmatrix} 0 & p_2\sigma^2 \\ -p_2\sigma^2 & 0\end{pmatrix} \pm \begin{pmatrix} 0 & p_3\sigma^3 \\ -p_3\sigma^3 & 0\end{pmatrix}\right\}$

Where from? Normalization?

$$= \frac{1}{2m}\begin{pmatrix} m\pm E & \mp\vec{p}\cdot\vec{\sigma} \\ \pm\vec{p}\cdot\vec{\sigma} & m\mp E\end{pmatrix}$$ using $p_0 = p^0 = E$ and $p_i = -p^i$ for spatial coordinates $\gamma^0 = \begin{pmatrix}\mathbb{1} & 0 \\ 0 & -\mathbb{1}\end{pmatrix}$, $\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0\end{pmatrix}$.

Now we will take a look at:

$$\sum_{r=1,2} u_r(p)\bar{u}_r(p) = u_1(p)u_1^\dagger(p)\gamma^0 + u_2(p)u_2^\dagger(p)\gamma^0$$

$$= \frac{E+m}{2m}\left\{\begin{pmatrix}\chi_1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_1\end{pmatrix}\left(\chi_1^T, \chi_1^T\left(\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\right)^\dagger\right) + \begin{pmatrix}\chi_2 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_2\end{pmatrix}\left(\chi_2^T, \chi_2^T\left(\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\right)^\dagger\right)\right\}\gamma^0$$

$\chi_1\chi_1^\dagger = \begin{pmatrix}1&0\\0&0\end{pmatrix}$ $\downarrow$ $\chi_2\chi_2^\dagger = \begin{pmatrix}0&0\\0&1\end{pmatrix}$

$$= \frac{E+m}{2m}\begin{pmatrix}\mathbb{1} & \left(\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\right)^\dagger \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} & \left(\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\right)\left(\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\right)^\dagger\end{pmatrix}\begin{pmatrix}\mathbb{1} & 0 \\ 0 & -\mathbb{1}\end{pmatrix}$$ using $\chi_1\chi_1^T + \chi_2\chi_2^T = \mathbb{1}$

$\vec{\sigma}^\dagger = \vec{\sigma}$, $\vec{p}^\dagger = \vec{p}$:
$$= \frac{1}{2m}\begin{pmatrix}E+m & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -\frac{(\vec{\sigma}\cdot\vec{p})^2}{E+m}\end{pmatrix} = \frac{1}{2m}\begin{pmatrix}E+m & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -\frac{\vec{p}^2}{E+m}\end{pmatrix} = \frac{1}{2m}\begin{pmatrix}E+m & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -\frac{E^2-m^2}{E+m}\end{pmatrix} = \frac{1}{2m}\begin{pmatrix}E+m & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & m-E\end{pmatrix}$$

Where we used that $(\vec{\sigma}\cdot\vec{p})^2 = \sigma^i p^i \sigma^j p^j = p^i p^j \sigma^i\sigma^j = p^i p^j\{\delta_{ij}\mathbb{1} + i\epsilon_{ijk}\sigma^k\} = \vec{p}^2 = E^2 - m^2$

as $i p^i p^j \epsilon_{ijk}\sigma^k = i\cdot\{\frac{1}{2}\epsilon_{ijk}p^i p^j\sigma^k + \frac{1}{2}\epsilon_{jik}p^j p^i\sigma^k\} = \frac{i}{2}\{\epsilon_{ijk}p^i p^j\sigma^k - \epsilon_{ijk}p^j p^i\sigma^k\} \underset{p^ip^j = p^jp^i}{=} 0$

Comparing those two yields $\boxed{\Lambda_+(p) = \sum_{r=1,2} u_r(p)\bar{u}_r(p)}$

Analogously, we get: $\sum_{r=1,2} v_r(p)\bar{v}_r(p) = v_1(p)v_1^\dagger(p)\gamma^0 + v_2(p)v_2^\dagger(p)\gamma^0$

$$= \frac{E+m}{2m}\left\{\begin{pmatrix}\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_1 \\ \chi_1\end{pmatrix}\left(\chi_1^T\left(\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\right)^\dagger, \chi_1^T\right) + \begin{pmatrix}\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_2 \\ \chi_2\end{pmatrix}\left(\chi_2^T\left(\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\right)^\dagger, \chi_2^T\right)\right\}\gamma^0$$

$$= \frac{E+m}{2m}\begin{pmatrix}\left(\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\right)^2 & \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} & \mathbb{1}\end{pmatrix}\begin{pmatrix}\mathbb{1} & 0 \\ 0 & -\mathbb{1}\end{pmatrix} = \frac{1}{2m}\begin{pmatrix}\frac{(\vec{\sigma}\cdot\vec{p})^2}{E+m} & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -(E+m)\end{pmatrix}$$

$$= -\frac{1}{2m}\begin{pmatrix}m-E & \vec{\sigma}\cdot\vec{p} \\ -\vec{\sigma}\cdot\vec{p} & m+E\end{pmatrix} = -\Lambda_-(p)$$ and thus what we claimed ☐

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$P \notin \mathcal{L}_+^\uparrow$?

H15) L.T: $x^\mu \longmapsto \Lambda^\mu{}_\nu x^\nu$, $\psi(x) \longmapsto S(\Lambda)\psi(x)$, $\bar\psi(x) \longmapsto \bar\psi(x) S^{-1}(\Lambda)$

What's the $\frac{i}{2}$ in $\sigma^{\mu\nu}$ for?

$\psi(\vec{x},t) \xmapsto{P} S_P \psi(-\vec{x},t)$, $S_P = e^{i\varphi}\gamma_0$, $\{\gamma_5, \gamma^\mu\} = 0$ (3) (lecture)

$S(\Lambda)^{-1}\gamma^\mu S(\Lambda) \overset{P12)}{=} \Lambda^\mu{}_\nu \gamma^\nu$ (1), $S(\Lambda)\gamma_5 \overset{\text{lecture}}{=} \gamma_5 S(\Lambda)$ (2)

$\psi(\vec{x},t) \xmapsto{P} S_P \psi(\vec{x},t)$

a) $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$

$$\bar\psi(x)\sigma_{\mu\nu}\psi(x) \longmapsto \bar\psi(x) S(\Lambda)^{-1} \frac{i}{2}[\gamma_\mu,\gamma_\nu] S(\Lambda)\psi(x)$$
$$= \frac{i}{2}\bar\psi(x) S(\Lambda)^{-1}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) S(\Lambda)\psi(x)$$
$$= \frac{i}{2}\Big\{\bar\psi(x) S(\Lambda)^{-1}\gamma_\mu S(\Lambda) S(\Lambda)^{-1}\gamma_\nu S(\Lambda)\psi(x) - \bar\psi(x) S(\Lambda)^{-1}\gamma_\nu S(\Lambda) S(\Lambda)^{-1}\gamma_\mu S(\Lambda)\psi(x)\Big\}$$

At what point did we use that $S(\Lambda) \in \mathcal{L}_+^\uparrow$? So should basically also hold for $S(\Lambda)=P$?

$$\overset{(1)}{=} \frac{i}{2}\Big\{\bar\psi(x)\Lambda^\sigma{}_\mu\gamma_\sigma \Lambda^\kappa{}_\nu \gamma_\kappa \psi(x) - \bar\psi(x)\Lambda^\kappa{}_\nu\gamma_\kappa\Lambda^\sigma{}_\mu\gamma_\sigma\psi(x)\Big\}$$
$$= \frac{i}{2}\Lambda^\sigma{}_\mu\Lambda^\kappa{}_\nu\Big\{\bar\psi(x)\gamma_\sigma\gamma_\kappa\psi(x) - \bar\psi(x)\gamma_\kappa\gamma_\sigma\psi(x)\Big\}$$
$$= \Lambda^\sigma{}_\mu\Lambda^\kappa{}_\nu\,\bar\psi(x)\frac{i}{2}[\gamma_\sigma,\gamma_\kappa]\psi(x)$$
$$= \Lambda^\sigma{}_\mu\Lambda^\kappa{}_\nu\,\bar\psi(x)\sigma_{\sigma\kappa}\psi(x)$$

$\bar\psi(x)\sigma_{\mu\nu}\psi(x) \xmapsto{P}$? ̈

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I Discounted 0.5 because You cannot forget that $\vec{x} \xmapsto{P} -\vec{x}$

b) $$\bar\psi(x)\gamma^\mu\gamma_5\psi(x) \longmapsto \bar\psi(x) S^{-1}(\Lambda)\gamma^\mu\gamma_5 S(\Lambda)\psi(x)$$
$$\overset{(2)}{=} \bar\psi(x) S(\Lambda)^{-1}\gamma^\mu S(\Lambda)\gamma_5\psi(x)$$
$$\overset{(1)}{=} \bar\psi(x)\Lambda^\mu{}_\nu\gamma^\nu\gamma_5\psi(x) = \Lambda^\mu{}_\nu\,\bar\psi(x)\gamma^\nu\gamma_5\psi(x)$$

Why only $\psi(x) \mapsto S_P\psi(x)$ and not $\psi(x) \mapsto S_P\psi(-x)$?

$$\bar\psi(x)\gamma^\mu\gamma_5\psi(x) \xmapsto{P} \bar\psi(-x) S_P^{-1}\gamma^\mu\gamma_5 S_P\psi(-x)$$

Remember $\bar\psi(\vec{x}) \xrightarrow{P} \bar\psi(-\vec{x})\gamma^0$

$$= \bar\psi(-x)\gamma_0\gamma^\mu\gamma_5\gamma_0\psi(-x) \overset{(3)}{=} -\bar\psi(-x)\gamma_0\gamma^\mu\gamma_0\gamma_5\psi(-x)$$
$$= \begin{cases} -\bar\psi(-x)\gamma_0\gamma_5\psi(-x), & \mu = 0 \\ -\bar\psi(-x)\gamma_\mu\gamma_5\psi(-x), & \mu \neq 0 \end{cases} \quad \text{as } \gamma^0 = \gamma_0,\ \gamma^i = -\gamma_i,\ \{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu} - \gamma^\nu\gamma^\mu$$

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Spatial reflection? Time reflection?

⇝ Pseudovector; Spatial reflection ⇝ time changes sign, spatial components don't

c) $\partial_\mu [\bar\psi(x) \gamma^\mu \gamma_5 \psi(x)] \longmapsto \Lambda_\mu{}^\kappa \partial_\kappa [\bar\psi(x) S(\Lambda)^{-1} \gamma^\mu \gamma_5 S(\Lambda) \psi(x)]$ Derivative also has to be transformed?

$\overset{(1)}{\underset{(2)}{=}} \Lambda^\kappa_\mu \partial_\kappa [\bar\psi(x) \Lambda^\mu_\nu \gamma^\nu \gamma_5 \psi(x)]$

$= \underbrace{\Lambda^\kappa_\mu \Lambda^\mu_\nu}_{=:\delta^\kappa_\nu} \partial_\kappa [\bar\psi(x) \gamma^\nu \gamma_5 \psi(x)]$ Why $\Lambda^\kappa_\mu \Lambda^\mu_\nu = \delta^\kappa_\nu$

$= \partial_\nu [\bar\psi(x) \gamma^\nu \gamma_5 \psi(x)]$

$\partial_\mu [\bar\psi(x) \gamma^\mu \gamma_5 \psi(x)] \longmapsto \tilde\partial_\mu [\bar\psi(x) S_P^{-1} \gamma^\mu \gamma_5 S_P \psi(x)]$, $\tilde\partial_\mu = \begin{pmatrix} \partial_0 \\ -\partial_1 \\ -\partial_2 \\ -\partial_3 \end{pmatrix}$

$= \tilde\partial_\mu [\bar\psi(x) \gamma_0 \gamma^\mu \gamma_5 \gamma_0 \psi(x)]$

$= \tilde\partial_\mu [-\bar\psi(x) \gamma_0 \gamma^\mu \gamma_0 \gamma_5 \psi(x)]$

$= -\tilde\partial_\mu [\bar\psi(x) \gamma_0 \gamma^\mu \gamma_0 \gamma_5 \psi(x)] = -\partial_\mu [\bar\psi(x) \gamma^\mu \gamma_5 \psi(x)]$

as $\gamma_0 \gamma^\mu \gamma_0 = \begin{cases} \gamma^0 & \text{for } \mu = 0 \\ -\gamma^\mu & \text{for } \mu \neq 0 \end{cases}$ Better way to write this down?

and $\tilde\partial_\mu \gamma^\mu = \partial_0 \gamma^0 - \partial_1 \gamma^1 - \partial_2 \gamma^2 - \partial_3 \gamma^3$

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$\rightsquigarrow$ Pseudo Scalar