Disclaimer

The solution at hand was written in the course of the respective class at the University of Bonn. If not stated differently on top of the first page or the following website, the solution was prepared and handed in solely by me, Marvin Zanke. Anything in a different color than the ball pen blue is usually a correction that I or a tutor made. For more information and all my material, check:
https://www.physics-and-stuff.com/

I raise no claim to correctness and completeness of the given solutions! This equally applies to the corrections mentioned above.

22.01.2017 Advanced Quantum Theory 11. Exercise

Marvin Zanke
Irina Kasabaci

H11b) $H = \vec{\alpha}\cdot\vec{p} + \beta m + e\phi$, $i\partial_t \psi = H\psi$ 20/20 Bravo!

Why U^+ if $U^+ = U$ and $UU^{-1} = 1$?

a) Looking for U (with $U^+ U = \mathbb{1}$) such that $H' = U H U^+$ is diagonal up to order $\mathcal{O}(m^{-1})$.

$\lambda^+ = \lambda$, $a^+ = a$?

Ansatz: $U = a\beta + \frac{\lambda}{m}\vec{\alpha}\cdot\vec{p} \rightsquigarrow U^+ = \beta^+ a^+ + \frac{\lambda^+}{m}\vec{p}^+\vec{\alpha}^+$

Why no $\vec{p}\cdot e\vec{A}$ term in our hamiltonian?

$$\mathbb{1} = UU^+ = \left(a\beta + \frac{\lambda}{m}\vec{\alpha}\cdot\vec{p}\right)\left(\beta a^+ + \frac{\lambda^+}{m}\vec{p}\cdot\vec{\alpha}\right) = a^2 + a\frac{\lambda}{m}\left\{\beta p^i\alpha^i + \alpha^i p^i \beta\right\} + \frac{\lambda^2}{m^2}\vec{p}^2$$

where we used that $(\vec{\alpha}\cdot\vec{p})(\vec{\alpha}\cdot\vec{p}) = \alpha^i p^i \alpha^j p^j = \alpha^i\alpha^j p^i p^j$
$= \frac{1}{2}\alpha^i\alpha^j p^i p^j + \frac{1}{2}\alpha^j\alpha^i p^j p^i$
$= \frac{1}{2}p^i p^j \{\alpha^i, \alpha^j\} = \vec{p}^2$

As $(\beta\alpha^i + \alpha^i\beta) = \{\beta, \alpha^i\} = 0$, it follows

Up to order $\mathcal{O}(m^{-1})$ → still m^{-1} terms included?

$$\mathbb{1} = UU^+ = a^2 + \frac{\lambda^2}{m^2}\vec{p}^2 \iff a^2 = \mathbb{1} + \cancel{\frac{\lambda^2}{m^2}\vec{p}^2} = \underbrace{\mathbb{1} + \mathcal{O}(m^{-2})}$$

Don't put it to 0 because you MAY encounter a $\mathcal{O}(m)$ term!!

Why ϕ of order $\mathcal{O}(m^{-1})$?

Now we calculate $H' = UHU^+ = U(\vec{\alpha}\cdot\vec{p} + \beta m + e\phi)U^+$

$$U\vec{\alpha}\cdot\vec{p}\,U^+ = \left(a\beta + \frac{\lambda}{m}\vec{\alpha}\cdot\vec{p}\right)\vec{\alpha}\cdot\vec{p}\left(a\beta + \frac{\lambda}{m}\vec{\alpha}\cdot\vec{p}\right)$$

using $\beta\alpha^i = -\alpha^i\beta$

$$= \underbrace{-a(\vec{\alpha}\cdot\vec{p})a}_{\mathcal{O}(m^0)\text{ and }\mathcal{O}(m^{-2})} + \underbrace{\frac{\lambda^2}{m^2}\cancel{(\vec{\alpha}\cdot\vec{p})^3}}_{\mathcal{O}(m^{-2})} + \frac{\lambda}{m}a\beta\underbrace{(\vec{\alpha}\cdot\vec{p})(\vec{\alpha}\cdot\vec{p})}_{\vec{p}^2} + \frac{\lambda}{m}\underbrace{(\vec{\alpha}\cdot\vec{p})(\vec{\alpha}\cdot\vec{p})}_{\vec{p}^2}a\beta$$

$$U\beta m U^+ = \left(a\beta + \frac{\lambda}{m}\vec{\alpha}\cdot\vec{p}\right)\beta m\left(a\beta + \frac{\lambda}{m}\vec{\alpha}\cdot\vec{p}\right)$$

Only order m^0 and m^{-2} and therefore also diagonal up to $\mathcal{O}(m^{-2})$?

$$= a m a\beta + \frac{\lambda^2}{m^2}\vec{\alpha}\cdot\vec{p}\,\beta m\,\vec{\alpha}\cdot\vec{p} + am\frac{\lambda}{m}\vec{\alpha}\cdot\vec{p} + \frac{\lambda}{m}\vec{\alpha}\cdot\vec{p}\,am$$

$$= ma^2\beta - \frac{\lambda^2}{m}\beta\underbrace{(\vec{\alpha}\cdot\vec{p})(\vec{\alpha}\cdot\vec{p})}_{\vec{p}^2} + \underbrace{2\lambda a(\vec{\alpha}\cdot\vec{p})}_{\mathcal{O}(m^0)\text{ and }\mathcal{O}(m^{-2})}$$

$$Ue\phi U^+ = \left(a\beta + \frac{\lambda}{m}\vec{\alpha}\cdot\vec{p}\right)e\phi\left(a\beta + \frac{\lambda}{m}\vec{\alpha}\cdot\vec{p}\right)$$

$$= ae\phi a + \frac{\lambda^2}{m^2}(\vec{\alpha}\cdot\vec{p})e\phi(\vec{\alpha}\cdot\vec{p}) + \underbrace{a\beta e\phi\frac{\lambda}{m}(\vec{\alpha}\cdot\vec{p})}_{\mathcal{O}(m^{-2})} + \underbrace{\frac{\lambda}{m}(\vec{\alpha}\cdot\vec{p})e\phi a\beta}_{\mathcal{O}(m^{-2})}$$

Now we see that odd powers of $\vec{\alpha}$ yield off-diagonal terms, while even powers result in diagonal terms. The off diagonal ones are underlined above with their order in m.

In leading order m^0, we therefore get for the off-diagonal terms:

$$-a(\vec{\alpha}\cdot\vec{p})a + 2\lambda a(\vec{\alpha}\vec{p}) \stackrel{!}{=} 0 \quad \text{for } \lambda = 1/2$$

and they therefore vanish.

$$a^2 = \mathbb{1} - \frac{1}{4m^2}\vec{p}^{\,2} \Leftrightarrow a = \sqrt{\mathbb{1} - \frac{1}{4m^2}\vec{p}^{\,2}} = 1 - \frac{\vec{p}^{\,2}}{8m^2} + \mathcal{O}(m^{-4})$$

You can't really expand like this since $\vec{p}$ could be $\gg m$

9/9

rotate rest mass away?

b) Using the ansatz $\psi(\vec r,t) = \begin{pmatrix} \chi(\vec r,t) \\ \eta(\vec r,t) \end{pmatrix} e^{-imt}$ where we seperated the leading energy dependence (rotated the rest mass away)

$$i\partial_t \psi(\vec r,t) = (\vec\alpha\cdot\vec p + \beta m + e\phi)\psi(\vec r,t)$$

Why $\mathcal{U}$ no effect on ∂_t?

$$\Leftrightarrow i\left\{-im\begin{pmatrix} \chi(\vec r,t) \\ \eta(\vec r,t) \end{pmatrix} e^{-imt} + \begin{pmatrix} \partial_t\chi(\vec r,t) \\ \partial_t\eta(\vec r,t) \end{pmatrix} e^{-imt}\right\} = (\vec\alpha\cdot\vec p + \beta m + e\phi)\begin{pmatrix} \chi(\vec r,t) \\ \eta(\vec r,t) \end{pmatrix} e^{-imt}$$

$$\Leftrightarrow i\partial_t \begin{pmatrix} \chi(\vec r,t) \\ \eta(\vec r,t) \end{pmatrix} = (\vec\alpha\cdot\vec p + m(\beta-1) + e\phi)\begin{pmatrix} \chi(\vec r,t) \\ \eta(\vec r,t) \end{pmatrix}$$

$\mathcal{U}$ on both sides and $\mathcal{U}^\dagger\mathcal{U} = \mathbb{1}$

$$\xLeftrightarrow{} \quad i\partial_t \begin{pmatrix} \chi'(\vec r,t) \\ \eta'(\vec r,t) \end{pmatrix} = \mathcal{U}(\vec\alpha\cdot\vec p + m(\beta-1) + e\phi)\mathcal{U}^\dagger \begin{pmatrix} \chi'(\vec r,t) \\ \eta'(\vec r,t) \end{pmatrix}$$

with $\begin{pmatrix} \chi'(\vec r,t) \\ \eta'(\vec r,t) \end{pmatrix} = \mathcal{U}\begin{pmatrix} \chi(\vec r,t) \\ \eta(\vec r,t) \end{pmatrix}$

Why H_{22}' only up to order $\mathcal{O}(m^{-1})$

Now we already constructed $\mathcal{U}H\mathcal{U}^\dagger$ s.t. off-diagonal terms vanish up to $\mathcal{O}(m^{-1})$

Taking a look at a) again and extracting the diagonal terms yields:

$$H_{11}' = \frac{1}{2m}a\,\vec p^2 + \frac{1}{2m}\vec p^2 a + ma^2 - \frac{1}{4m}\vec p^2 + ae\phi a + \frac{1}{4m^2}(\vec\sigma\cdot\vec p)e\phi(\vec\sigma\cdot\vec p) - m$$

new from rotat.

$$H_{22}' = -\frac{1}{2m}a\vec p^2 - \frac{1}{2m}\vec p^2 a - ma^2 + \frac{1}{4m}\vec p^2 + ae\phi a + \frac{1}{4m^2}(\vec\sigma\cdot\vec p)e\phi(\vec\sigma\cdot\vec p) - m$$

where we used $\beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$ and

$$(\vec\alpha\cdot\vec p)(\vec\alpha\cdot\vec p) = \alpha^i p^i \alpha^j p^j = \alpha^i\alpha^j p^i p^j = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^j \\ \sigma^j & 0 \end{pmatrix} p^i p^j$$
$$= \begin{pmatrix} \sigma^i\sigma^j & 0 \\ 0 & \sigma^i\sigma^j \end{pmatrix} p^i p^j = \sigma^i p^i \sigma^j p^j\,\mathbb{1}$$

Why only interested in $H_{11}' \mathrel{\hat=} H_{eff}$ and not H_{22}'?

$$i\partial_t \chi' = H_{11}'\chi' + \mathcal{O}(\ )$$
$$i\partial_t \eta' = H_{22}'\eta' + \mathcal{O}(\ )$$

Why in H_{22}' only up to $\mathcal{O}(m^{-1})$?

Now: $H_{22}' = -2m + \mathcal{O}(m^{-1})$

We are interested in $H_{eff} \equiv H_{11}'\big|_{\lambda=1/2}$

$$H_{11}' = \frac{1}{2m}\vec p^2 - \frac{1}{2m}\frac{\vec p^4}{8m^2} + \frac{1}{2m}\vec p^2 - \frac{1}{2m}\frac{\vec p^4}{8m^2} + m - \frac{\vec p^2}{4m} - \frac{\vec p^2}{4m} + e\phi - \frac{\vec p^2}{8m^2}e\phi$$
$$- e\phi\frac{\vec p^2}{8m^2} + \frac{1}{4m^2}(\vec\sigma\cdot\vec p)e\phi(\vec\sigma\cdot\vec p) - \cancel{m}$$

$$\Rightarrow H'_{11} = \frac{\vec{p}^2}{2m} - \frac{p^4}{8m^3} + e\phi \underbrace{- \frac{\vec{p}^2}{8m^2} e\phi}_{(1)} - e\phi \frac{\vec{p}^2}{8m^2} + \underbrace{\frac{1}{4m^2}(\vec{\sigma}\cdot\vec{p})\, e\phi\, (\vec{\sigma}\cdot\vec{p})}_{(2)}$$

$$(1) = -\frac{e}{8m^2}\vec{p}\vec{p}\phi = -\frac{e}{8m^2}\vec{p}\left\{[\vec{p}\phi] + \phi\vec{p}\right\} = -\frac{e}{8m^2}\left\{[\vec{p}^2\phi] + 2[\vec{p}\phi]\vec{p} + \phi\vec{p}^2\right\}$$

$$(2) = \frac{e}{4m^2}\left\{(\vec{\sigma}\cdot\vec{p})\phi(\vec{\sigma}\cdot\vec{p})\right\} = \frac{e}{4m^2}\left\{\sigma^i p^i \phi \sigma^j p^j\right\} = \frac{e}{4m^2}\left\{\sigma^i\sigma^j p^i \phi p^j\right\}$$

$$= \frac{e}{4m^2}\left\{(\delta^{ij} + i\epsilon_{ijk}\sigma^k) p^i\phi p^j\right\} = \frac{e}{4m^2}\left\{(\delta^{ij} + i\epsilon_{ijk}\sigma^k)((p^i\phi)p^j + \phi p^i p^j)\right\}$$

$\epsilon_{ijk}\sigma^k p^i p^j = 0$ as antisym. $=$

$$\frac{e}{4m^2}\left\{[\vec{p}\phi]\vec{p} + i\epsilon_{ijk}(p^i\phi)p^j\sigma^k + \phi\vec{p}^2\right\} = \frac{e}{4m^2}\left\{[\vec{p}\phi]\vec{p} + i([\vec{p}\phi]\times\vec{p})\vec{\sigma} + \phi\vec{p}^2\right\}$$

$$\rightsquigarrow H'_{11} = \frac{\vec{p}^2}{2m} + e\phi - \frac{p^4}{8m^3} \underbrace{- \frac{e}{8m^2}[\vec{p}^2\phi]}_{(3)} + \underbrace{i\frac{e}{4m^2}([\vec{p}\phi]\times\vec{p})\vec{\sigma}}_{(4)}$$

with $\vec{p} = -i\vec{\nabla}$ we get:

$$(3) = -\frac{e}{8m^2}(-\vec{\nabla}^2\phi) = \frac{e}{8m^2}(\vec{\nabla}^2\phi)$$

$$(4) = \frac{e}{4m^2}\vec{\sigma}([\vec{\nabla}\phi]\times\vec{p}) = \frac{e}{4m^2}\vec{\sigma}((\vec{e}_r\,\partial\phi)\times\vec{p}) = \frac{e}{4m^2}\vec{\sigma}\left(\frac{1}{r}\frac{\partial\phi}{\partial r}\vec{L}\right)$$

$\phi = \phi(r)$ and $\vec{\nabla} = \vec{e}_r\,\partial_r + \vec{e}_\theta\frac{1}{r}\partial_\theta + \vec{e}_\varphi\frac{1}{r\sin\theta}\partial_\varphi$

using $\vec{L} = \vec{r}\times\vec{p}$ and $\vec{e}_r = \frac{\vec{r}}{r}$

$$\rightsquigarrow H'_{11} = \frac{\vec{p}^2}{2m} + e\phi - \frac{p^4}{8m^3} + \frac{e}{8m^2}(\vec{\nabla}^2\phi) + \frac{e}{4m^2}\frac{1}{r}\frac{\partial\phi}{\partial r}(\vec{\sigma}\cdot\vec{L})$$

NICE! 8/8

What equations for χ' and η'?

c) We want to calculate

$$\langle H_{eff} \rangle = \langle H_{non-rel} + \tilde{H} \rangle \quad \text{with} \quad \tilde{H} = -\frac{\vec{p}^{\,4}}{8m^3} + \frac{e}{8m^2}[\vec{\nabla}^2\phi] + \frac{e}{4m^2}\frac{1}{r}\frac{d\phi}{dr}(\vec{\sigma}\vec{L})$$

$$H_{non-rel} = \frac{\vec{p}^{\,2}}{2m} + e\phi$$

and $\langle \tilde{H} \rangle = \Delta E_{nl}^{rel}$

$\langle H_{non-rel} \rangle = E_{nl}$

$$\langle \tilde{H} \rangle = \langle -\frac{\vec{p}^{\,4}}{8m^3} \rangle + \langle \frac{e}{8m^2}[\vec{\nabla}^2\phi] \rangle + \langle \frac{e}{4m^2}\frac{1}{r}\frac{d\phi}{dr}(\vec{\sigma}\vec{L}) \rangle$$

only $l=0$ is $\neq 0$ at origin?

$$= -\frac{1}{8m^3}\langle \vec{p}^{\,4} \rangle + \underbrace{\frac{e}{8m^2}\langle [\vec{\nabla}^2\phi] \rangle}_{\text{only contributes for } l=0} + \underbrace{\frac{e}{4m^2}\langle \frac{1}{r}\frac{d\phi}{dr}(\vec{\sigma}\vec{L}) \rangle}_{\text{only contributes for } l\neq 0}$$

Factor $\frac{1}{4\pi}$?

with $\phi = -\frac{e}{4\pi r}$, $\vec{\nabla}\phi = -\vec{E} \Leftrightarrow \vec{\nabla}^2\phi = -\vec{\nabla}\vec{E}$

$\vec{\nabla}\vec{E} = \rho = -e\delta(r)$, $\frac{d\phi}{dr} = \frac{e}{4\pi r^2}$, $\alpha := \frac{e^2}{4\pi}$

we get two cases:

Why do $\langle 1/r^3 \rangle$ and $\langle \vec{\sigma}\vec{L} \rangle$ split?

$\underline{l \neq 0}$: $$\langle \tilde{H} \rangle = -\frac{1}{8m^3}\left(4m^4\alpha^4\frac{1}{n^4}\left(\frac{n}{l+1/2} - 3/4\right)\right) + \frac{e^2}{4m^2}\langle \frac{1}{4\pi r^3}\vec{\sigma}\vec{L} \rangle$$

$$= -\frac{m}{2}\frac{\alpha^4}{n^4}\left(\frac{n}{l+1/2} - 3/4\right) + \frac{1}{4\pi}\frac{e^2}{4m^2}\left(\frac{m^3\alpha^3}{n^3 l(l+1/2)(l+1)}\right)\left(j(j+1) - 3/4 - l(l+1)\right)$$

$\alpha = e^2$

$$= -\frac{m}{2}\frac{\alpha^4}{n^4}\left(\frac{n}{l+1/2} - 3/4\right) + \frac{m}{2}\frac{\alpha^4}{2n^3}\frac{j(j+1) - l(l+1) - 3/4}{l(l+1/2)(l+1)}$$

$$= -\frac{m}{2}\frac{\alpha^4}{n^4}\left(\frac{n}{l+1/2} - \frac{n}{2}\frac{j(j+1) - l(l+1) - 3/4}{l(l+1/2)(l+1)} - 3/4\right)$$

$j = l+1/2$ or also $j = l-1/2$

$$= \begin{cases} -\frac{m}{2}\frac{\alpha^4}{n^4}\left(\frac{2nl(l+1) - n(l+1/2)(l+3/2) + nl(l+1) + 3/4 n}{2(l+1/2)(l+1)l} - 3/4\right), & j = l+1/2 \\ -\frac{m}{2}\frac{\alpha^4}{n^4}\left(\frac{2nl(l+1) - n(l-1/2)(l+1/2) + nl(l+1) + 3/4 n}{2l(l+1)(l+1/2)} - 3/4\right), & j = l-1/2 \end{cases}$$

$$= \begin{cases} -\frac{m}{2}\frac{\alpha^4}{n^4}\left(\frac{2nl^2 + 2nl - n(l^2 + 2l + 3/4) + nl^2 + nl + 3/4 n}{2l(l+1)(l+1/2)} - 3/4\right), & j = l+1/2 \\ -\frac{m}{2}\frac{\alpha^4}{n^4}\left(\frac{2nl^2 + 2nl - n(l^2 - 1/4) + nl^2 + nl + 3/4 n}{2l(l+1)(l+1/2)} - 3/4\right), & j = l-1/2 \end{cases}$$

$$= \begin{cases} -\frac{m}{2}\frac{\alpha^4}{n^4}\left(\frac{2nl^2 + nl}{2l(l+1)(l+1/2)} - 3/4\right), & j = l+1/2 \\ -\frac{m}{2}\frac{\alpha^4}{n^4}\left(\frac{2nl^2 + 3nl + n}{2l(l+1)(l+1/2)} - 3/4\right), & j = l-1/2 \end{cases}$$

$$= \begin{cases} -\frac{m}{2}\frac{\alpha^4}{n^4}\left(n\frac{l+1/2}{(l+1)(l+1/2)} - 3/4\right), & j=l+1/2 \\ -\frac{m}{2}\frac{\alpha^4}{n^4}\left(n\frac{2l^2+2l+l+1}{2l(l+1)(l+1/2)} - 3/4\right), & j=l-1/2 \end{cases} = \begin{cases} -\frac{m}{2}\frac{\alpha^4}{n^4}\left(n\frac{1}{l+1} - \frac{3}{4}\right), & j=l+1/2 \\ -\frac{m}{2}\frac{\alpha^4}{n^4}\left(n\frac{2l+1}{2l(l+1/2)} - 3/4\right), & j=l-1/2 \end{cases}$$

$$= \begin{cases} -\frac{m}{2}\frac{\alpha^4}{n^4}\left(\frac{n}{l+1} - \frac{3}{4}\right), & j=l+1/2 \\ -\frac{m}{2}\frac{\alpha^4}{n^4}\left(\frac{n}{l} - \frac{3}{4}\right), & j=l-1/2 \end{cases} = -\frac{m}{2}\frac{\alpha^4}{n^4}\left(\frac{n}{j+1/2} - \frac{3}{4}\right)$$

$\underline{l=0}$: $$\langle \hat{H}' \rangle = -\frac{1}{8m^3}\left(4m^4\alpha^4\frac{1}{n^4}\left(\frac{n}{l+1/2} - \frac{3}{4}\right)\right)\Big|_{l=0} - \frac{e}{8m^2}\langle \vec{\nabla}\vec{E}\rangle$$

$$= -\frac{m}{2}\frac{\alpha^4}{n^4}\left(2n - 3/4\right) + \frac{e^2}{8m^2}\langle \delta^{(3)}(r)\rangle$$

$$= -\frac{m}{2}\frac{\alpha^4}{n^4}\left(2n - 3/4\right) + \frac{e^2}{8m^2}|\psi_{n00}(0)|^2$$

Integration $\int dr\, r^2 \psi^2 \delta(r) = 0$ because of r^2?

$|\psi_{n00}(0)|^2 = \frac{m^3\alpha^3}{\pi n^3}$ hint from sheet

$$= -\frac{m}{2}\frac{\alpha^4}{n^4}\left(2n - 3/4\right) + \frac{e^2}{8m^2}\frac{m^3\alpha^3}{\pi n^3}$$

$$= -\frac{m}{2}\frac{\alpha^4}{n^4}\left(2n - 3/4\right) + \frac{m}{2}\frac{\alpha^4}{n^3}$$

$$= -\frac{m}{2}\frac{\alpha^4}{n^4}\left(2n - 3/4 - n\right) = -\frac{m}{2}\frac{\alpha^4}{n^4}\left(n - 3/4\right)$$

$$= -\frac{m}{2}\frac{\alpha^4}{n^4}\left(\frac{n}{j+1/2} - \frac{3}{4}\right)$$

3/3