Disclaimer

The solution at hand was written in the course of the respective class at the University of Bonn. If not stated differently on top of the first page or the following website, the solution was prepared and handed in solely by me, Marvin Zanke. Anything in a different color than the ball pen blue is usually a correction that I or a tutor made. For more information and all my material, check:
https://www.physics-and-stuff.com/

I raise no claim to correctness and completeness of the given solutions! This equally applies to the corrections mentioned above.

01.02.2017 Advanced Quantum Theory 13. Exercise

Marvin Zanke
Trim Kasabeci

H19*)

$$V_2 = \frac{1}{2} \sum_{ijkl} \langle i,j|V|k,l\rangle a_i^\dagger a_j^\dagger a_l a_k$$

where from? →

$$\langle i,j|V|k,l\rangle = \int d^3x\, d^3x'\, \varphi_i^*(\vec{x})\, \varphi_j^*(\vec{x}')\, V(\vec{x},\vec{x}')\, \varphi_k(\vec{x})\, \varphi_l(\vec{x}')$$

$$\psi(\vec{x}) := \sum_i \varphi_i(\vec{x})\, a_i \qquad \psi^\dagger(\vec{x}) := \sum_i \varphi_i^*(\vec{x})\, a_i^\dagger$$

a)

$$V_2 = \frac{1}{2}\int d^3x\, d^3x'\, \psi^\dagger(\vec{x})\, \psi^\dagger(\vec{x}')\, V(\vec{x},\vec{x}')\, \psi(\vec{x}')\, \psi(\vec{x})$$

$$= \frac{1}{2}\int d^3x\, d^3x' \left(\sum_i \varphi_i^*(\vec{x})\, a_i^\dagger\right)\left(\sum_j \varphi_j^*(\vec{x}')\, a_j^\dagger\right) V(\vec{x},\vec{x}') \left(\sum_k \varphi_k(\vec{x}')\, a_k\right)\left(\sum_l \varphi_l(\vec{x})\, a_l\right)$$

$a_{j/i}^{(\dagger)}$ no effect on φ_i and $V(\vec{x},\vec{x}')$

$$= \frac{1}{2}\sum_{ijkl} \int d^3x\, d^3x'\, \varphi_i^*(\vec{x})\, a_i^\dagger\, \varphi_j^*(\vec{x}')\, a_j^\dagger\, V(\vec{x},\vec{x}')\, \varphi_k(\vec{x}')\, a_k\, \varphi_l(\vec{x})\, a_l$$

$$= \frac{1}{2}\sum_{ijkl} \int d^3x\, d^3x'\, \varphi_i^*(\vec{x})\, \varphi_j^*(\vec{x}')\, V(\vec{x},\vec{x}')\, \varphi_k(\vec{x}')\, \varphi_l(\vec{x})\, a_i^\dagger a_j^\dagger a_k a_l$$

$$= \frac{1}{2}\sum_{ijkl} \underbrace{\int d^3x\, d^3x'\, \varphi_i^*(\vec{x})\, \varphi_j^*(\vec{x}')\, V(\vec{x},\vec{x}')\, \varphi_l(\vec{x})\, \varphi_k(\vec{x}')}_{\langle i,j|V|l,k\rangle}\, a_i^\dagger a_j^\dagger a_k a_l$$

renaming k, l

$$= \frac{1}{2}\sum_{ijkl} \langle i,j|V|k,l\rangle\, a_i^\dagger a_j^\dagger a_l a_k$$

b) next page ...

Why cont. momentum spectrum? In contrary to $\psi(\vec{x}) = \sum_i \varphi_i(\vec{x}) a_i$?

(using $\psi(\vec{x}) = \int d^3k \frac{1}{(2\pi)^3} \varphi_{\vec{k}}(\vec{x})\, a_{\vec{k}}$, $\varphi_{\vec{k}}(\vec{x}) = e^{i\vec{k}\vec{x}}$

$\psi^\dagger(\vec{x}) = \int d^3k \frac{1}{(2\pi)^3} \varphi_{\vec{k}}^*(\vec{x})\, a_{\vec{k}}^\dagger$, $\varphi_{\vec{k}}^*(\vec{x}) = e^{-i\vec{k}\vec{x}}$

$[a_{\vec{k}}, a_{\vec{k}'}] = [a_{\vec{k}}^\dagger, a_{\vec{k}'}^\dagger] = 0$

$[a_{\vec{k}}, a_{\vec{k}'}^\dagger] = (2\pi)^3\, \delta^{(3)}(\vec{k}-\vec{k}')$)

b) We start off at (13) with the result and bring it back to the form (10).
We will be using the following representation of the δ-Distribution: $\delta(x-a) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(x-a)t}\,dt$ in dimension 3.

$$= \delta(a-x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i(x-a)t}\,dt \quad (*)$$

We therefore get:

$$V_2 = \int \frac{d^3p'}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\frac{1}{2}\overbrace{V(\vec p\,'-\vec p)}^{\int d^3x\, e^{-i(\vec p\,'-\vec p)\vec x} V(\vec x) \text{ Fourier-transform}} a^\dagger_{\vec p\,'} a^\dagger_{\vec k'} a_{\vec k} a_{\vec p}\,\overbrace{(2\pi)^3\delta^{(3)}(\vec k' + \vec p\,' - \vec p - \vec k)}^{(*)}$$

$$= \frac{1}{2}\int \frac{d^3p'}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\int d^3x\, e^{-i(\vec p\,'-\vec p)\vec x} V(\vec x)\, a^\dagger_{\vec p\,'} a^\dagger_{\vec k'} a_{\vec k} a_{\vec p}\int d^3x'\, e^{-i(\vec k'+\vec p\,'-\vec p-\vec k)\vec x'}$$

$\vec x\,'' := \vec x + \vec x\,'$, $\frac{dx''}{dx^{(i)}} = 1$

$$= \frac{1}{2}\int \frac{d^3p'}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\int d^3x'' d^3x'\, e^{-i(\vec p\,'-\vec p)(\vec x\,''-\vec x\,')} V(\vec x\,''-\vec x\,') e^{-i(\vec k'+\vec p\,'-\vec p-\vec k)\vec x\,'} a^\dagger_{\vec p\,'} a^\dagger_{\vec k'} a_{\vec k} a_{\vec p}$$

$$= \frac{1}{2}\int \frac{d^3p'}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\int d^3x'' d^3x'\, e^{-i(\vec p\,'-\vec p)\vec x\,''} V(\vec x\,''-\vec x\,') e^{-i(\vec k'-\vec k)\vec x\,'} a^\dagger_{\vec p\,'} a^\dagger_{\vec k'} a_{\vec k} a_{\vec p}$$

$\vec x\,'' \equiv \vec x$ renaming

$$= \frac{1}{2}\int \frac{d^3p'}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\int d^3x\, d^3x'\, e^{-i(\vec p\,'-\vec p)\vec x} V(\vec x-\vec x\,') e^{-i(\vec k'-\vec k)\vec x\,'} a^\dagger_{\vec p\,'} a^\dagger_{\vec k'} a_{\vec k} a_{\vec p}$$

No commutation relation needed?

$$= \frac{1}{2}\int \frac{d^3p'}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\int d^3x\, d^3x'\, \underbrace{e^{-i\vec p\,'\vec x}}_{\varphi^*_{\vec p\,'}(\vec x)} a^\dagger_{\vec p\,'} \underbrace{e^{-i\vec k'\vec x\,'}}_{\varphi^*_{\vec k'}(\vec x\,')} a^\dagger_{\vec k'} V(\vec x-\vec x\,') \underbrace{e^{i\vec k\vec x\,'}}_{\varphi_{\vec k}(\vec x\,')} a_{\vec k} \underbrace{e^{i\vec p\vec x}}_{\varphi_{\vec p}(\vec x)} a_{\vec p}$$

$$= \frac{1}{2}\int d^3x\, d^3x' \underbrace{\left(\int\frac{d^3p'}{(2\pi)^3}\varphi^*_{\vec p\,'}(\vec x) a^\dagger_{\vec p\,'}\right)}_{\psi^\dagger(\vec x)}\underbrace{\left(\int\frac{d^3k'}{(2\pi)^3}\varphi^*_{\vec k'}(\vec x\,') a^\dagger_{\vec k'}\right)}_{\psi^\dagger(\vec x\,')} V(\vec x-\vec x\,')$$

$$\times \underbrace{\left(\int\frac{d^3k}{(2\pi)^3}\varphi_{\vec k}(\vec x\,') a_{\vec k}\right)}_{\psi(\vec x\,')}\underbrace{\left(\int\frac{d^3p}{(2\pi)^3}\varphi_{\vec p}(\vec x) a_{\vec p}\right)}_{\psi(\vec x)}$$

Change position of $V(\vec x-\vec x\,')$ to front of integral? ?

$$= \frac{1}{2}\int d^3x\, d^3x'\, \psi^\dagger(\vec x)\psi^\dagger(\vec x\,') V(\vec x-\vec x\,')\psi(\vec x\,')\psi(\vec x) \quad \blacksquare$$

Annihilating particles with momentum $\vec k$ and $\vec p$
Creating particles with momentum $\vec k'$ and $\vec p\,'$ $\Big\}$ $\rightsquigarrow$ transfer of momentum
$\delta^{(3)}$ – for momentum conservation