Disclaimer

The solution at hand was written in the course of the respective class at the University of Bonn. If not stated differently on top of the first page or the following website, the solution was prepared and handed in solely by me, Marvin Zanke. Anything in a different color than the ball pen blue is usually a correction that I or a tutor made. For more information and all my material, check:
https://www.physics-and-stuff.com/

I raise no claim to correctness and completeness of the given solutions! This equally applies to the corrections mentioned above.

(H20*)

$n(\vec{x}) = \sum_\lambda \psi^+_\lambda(\vec{x})\,\psi_\lambda(\vec{x})$ | $\psi_\lambda(\vec{x}) = \sum_{\vec{k}} \varphi_{\vec{k}}(\vec{x})\, a_{\vec{k},\lambda}$, $\psi^+_\lambda(\vec{x}) = \sum_{\vec{k}} \varphi^*_{\vec{k}}(\vec{x})\, a^+_{\vec{k},\lambda}$ $\left\{ \varphi_{\vec{k}}(\vec{x}) = \frac{1}{\sqrt{V}} e^{i\vec{k}\vec{x}},\quad N = \int d^3x\, n(\vec{x}) \right.$

mom. repr. of FT ?

a)

$$n_{\vec{q}} = \int d^3x\, n(\vec{x})\, e^{-i\vec{q}\vec{x}} = \int d^3x \sum_\lambda \psi^+_\lambda(\vec{x})\,\psi_\lambda(\vec{x})\, e^{-i\vec{q}\vec{x}}$$

$$= \int d^3x \sum_{\lambda,\vec{k},\vec{k}'} \left(\varphi^*_{\vec{k}}(\vec{x})\, a^+_{\vec{k},\lambda}\right)\left(\varphi_{\vec{k}'}(\vec{x})\, a_{\vec{k}',\lambda}\right) e^{-i\vec{q}\vec{x}}$$

$$= \sum_{\vec{k},\vec{k}',\lambda} a^+_{\vec{k},\lambda}\, a_{\vec{k}',\lambda} \int d^3x\, \varphi^*_{\vec{k}}(\vec{x})\, \varphi_{\vec{k}'}(\vec{x})\, e^{-i\vec{q}\vec{x}}$$

$$= \sum_{\vec{k},\vec{k}',\lambda} a^+_{\vec{k},\lambda}\, a_{\vec{k}',\lambda} \underbrace{\int \frac{1}{V} e^{-i\vec{k}\vec{x}}\, e^{i\vec{k}'\vec{x}}\, e^{-i\vec{q}\vec{x}}\, d^3x}$$

$$\boxed{\frac{1}{V}\int d^3x\, e^{i\vec{x}(\vec{k}'-\vec{k}-\vec{q})} = \frac{(2\pi)^3}{V}\,\delta(\vec{k}' - (\vec{k}+\vec{q}))}$$

$\boxed{(2\pi)^3 = V\ ?}$?

$P_s\varphi = \beta\varphi$

$$= \sum_{\vec{k},\lambda} \frac{(2\pi)^3}{V}\, a^+_{\vec{k},\lambda}\, a_{\vec{k}+\vec{q},\lambda}$$

b)

$$S(\vec{q}) := \frac{1}{N} \langle F | n_{\vec{q}}\, n_{-\vec{q}} | F \rangle$$

$$\vec{q} = 0 \Rightarrow \quad S(0) = \frac{1}{N}\langle F | n_0 n_0 | F\rangle = \frac{1}{N}\langle F | \Big(\sum_{\vec{k},\lambda} a^+_{\vec{k},\lambda} a_{\vec{k},\lambda}\Big)\Big(\sum_{\vec{k}',\lambda'} a^+_{\vec{k}',\lambda'} a_{\vec{k}',\lambda'}\Big) | F\rangle$$

$$= \frac{1}{N}\langle F | \sum_{\vec{k},\vec{k}',\lambda,\lambda'} a^+_{\vec{k},\lambda} a_{\vec{k},\lambda}\, a^+_{\vec{k}',\lambda'} a_{\vec{k}',\lambda'} | F\rangle$$

Static struct. fct. for non int. acting fermions?

?

$$= \frac{1}{N}\Big/ \langle F | \sum_{\vec{k},\lambda} a^+_{\vec{k},\lambda} a_{\vec{k},\lambda} | \tilde{F}\rangle \langle \tilde{F} | \sum_{\vec{k}',\lambda'} a^+_{\vec{k}',\lambda'} a_{\vec{k}',\lambda'} | F\rangle$$

$$= \frac{1}{N}\Big/ \sum_{\substack{\vec{k},\lambda\\ \vec{k}',\lambda'}} \langle F | a^+_{\vec{k},\lambda} a_{\vec{k},\lambda} | \tilde{F}\rangle \langle \tilde{F} | a^+_{\vec{k}',\lambda'} a_{\vec{k}',\lambda'} | F\rangle$$

$$= \frac{1}{N} N^2 = N$$

$$\frac{1}{N} \sum_{\substack{\vec{k},\lambda\\ \vec{k}',\lambda'}} \langle F | n_{\vec{k},\lambda}\, n_{\vec{k}',\lambda'} | F\rangle$$

$$= \frac{1}{N} N^2 = N$$

$\vec{k}' = \vec{k}$ and $\lambda = \lambda$ to fill all states where holes were left !

c) $S^0(\vec{q}) = \frac{1}{N} \langle F | n_{\vec{q}} \, n_{-\vec{q}} | F \rangle$

$$= \frac{1}{N} \langle F | \Big(\sum_{\vec{k},\lambda} a^+_{\vec{k},\lambda} a_{\vec{k}+\vec{q},\lambda} \Big) \Big(\sum_{\vec{k}',\lambda'} a^+_{\vec{k}',\lambda'} a_{\vec{k}'-\vec{q},\lambda'} \Big) | F \rangle$$

$$= \frac{1}{N} \langle F | \sum_{\substack{\vec{k},\lambda \\ \vec{k}',\lambda'}} a^+_{\vec{k},\lambda} a_{\vec{k}+\vec{q},\lambda} a^+_{\vec{k}',\lambda'} a_{\vec{k}'-\vec{q},\lambda'} | F \rangle$$

$$\{ a_{\vec{m}}, a^+_{\vec{n}} \} = \delta_{mn}$$

$$\left.\begin{array}{l} \lambda', \vec{k}' - \vec{q} = \vec{k}' \\ \lambda, \vec{k} + \vec{q} = \vec{k} \end{array}\right\} \vec{q} = 0$$

$$\vec{k}' - \vec{q}, \lambda' = \vec{k}, \lambda$$

$$\vec{k} + \vec{q}, \lambda = \vec{k}', \lambda'$$

$$= \frac{1}{N} \sum_{\substack{\vec{k},\lambda \\ \vec{k}',\lambda'}} \delta_{\vec{k}\vec{k}'-\vec{q}} \, \delta_{\lambda\lambda'} \langle F | a^+_{\vec{k},\lambda} a_{\vec{k}'-\vec{q},\lambda'} \big(\delta_{\vec{k}+\vec{q},\vec{k}'} \delta_{\lambda\lambda'} - a^+_{\vec{k}',\lambda'} a_{\vec{k}+\vec{q},\lambda} \big) | F \rangle$$

$a^+_{\vec{k},\lambda} a_{\vec{k},\lambda} \overset{?}{=} n_{\vec{k},\lambda}$?

$$= \frac{1}{N} \sum_{\vec{k},\lambda} \langle F | a^+_{\vec{k},\lambda} a_{\vec{k},\lambda} \big(1 - a^+_{\vec{k}+\vec{q},\lambda} a_{\vec{k}+\vec{q},\lambda} \big) | F \rangle$$

$$= \frac{1}{N} \sum_{\vec{k},\lambda} \Big\{ \langle F | n_{\vec{k},\lambda} | F \rangle - \langle F | n_{\vec{k}+\vec{q},\lambda} n_{\vec{k}+\vec{q},\lambda} | F \rangle \Big\}$$

$$= 1 - \frac{1}{N} \langle F | n_{\vec{k}+\vec{q},\lambda} n_{\vec{k}+\vec{q},\lambda} | F \rangle$$

d) $S^0(\vec{q}) = 1 - \frac{1}{N} \sum_{\vec{k},\lambda} \langle F | n_{\vec{k},\lambda} n_{\vec{k}+\vec{q},\lambda} | F \rangle$

$$\approx 1 - \frac{2V^2}{(2\pi)^6 N} \int d^3k\, d^3q\; \theta(k_F - |\vec{k}|)\, \theta(\underbrace{k_F - |\vec{k}+\vec{q}|}_{})$$

$$\theta\left(k_F - \sqrt{|\vec{k}|^2 + |\vec{q}|^2 + 2\vec{q}\cdot\vec{k}}\right)$$

Need a) for this?

1 case: $0 < q \leq k_F$

$$\left(\frac{d\vec{z}}{d\vec{q}} = 1 \;\middle|\; \vec{z} := \vec{k}+\vec{q} \right)$$

Possible to do substitution like this?

$$\left(1 - \frac{2V^2}{(2\pi)^6 N} \int d^3k\, d^3z\; \theta(k_F - |\vec{k}|)\, \theta(k_F - |\vec{z}|) \right)$$

$\vec{k} + \frac{\vec{q}}{2} =: \tilde{\vec{k}}$

$$1 - \frac{2V^2}{(2\pi)^6 N} \int d^3\tilde{k} \int d^3q\; \theta(k_F - |\tilde{\vec{k}} - \tfrac{\vec{q}}{2}|) \times \theta(k_F - |\tilde{\vec{k}} + \tfrac{\vec{q}}{2}|)$$

$$\frac{d\tilde{\vec{k}}}{d\vec{k}} = 1 \qquad \frac{d\tilde{\vec{k}}}{d\vec{q}} = \frac{1}{2}$$

$k_F \geq |\tilde{\vec{k}} - \frac{\vec{q}}{2}|$

1) $k_F \geq \tilde{\vec{k}} - \frac{\vec{q}}{2}$

2) $k_F \leq -\tilde{\vec{k}} + \frac{\vec{q}}{2}$

$k_F \geq \tilde{\vec{k}} + \frac{\vec{q}}{2}$

$k_F \leq -\tilde{\vec{k}} - \frac{\vec{q}}{2}$