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Advanced Quantum theory Exercise 2 Marvin Zanke

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H3)

a) Given $V(\vec{r}) = V_0 \frac{e^{-\mu r}}{r}$, we want to calculate its Fourier transform - the Yukawa Potential in momentum space

Why don't we need the prefactor of the Fourier-transform here? And is it exp{-...} or exp{...}?

Our convention: with "-" and without prefactor to momentum

$$V(\vec{p}) = \int d^3x \; V(\vec{x}) \, e^{-\frac{i\vec{p}\vec{x}}{\hbar}}$$

$$= \int_0^\infty dr \int_0^\pi d\theta \int_0^{2\pi} d\varphi \; V(r) \, e^{-\frac{ipr\cos\theta}{\hbar}} \, r^2 \sin\theta$$

$$= \int_0^\infty dr \int_0^\pi d\theta \int_0^{2\pi} d\varphi \; V_0 \frac{e^{-\mu r - ipr\cos\theta/\hbar}}{r} r^2 \sin\theta = 2\pi V_0 \int_0^\infty dr \int_0^\pi d\theta \; e^{-\mu r} \cdot r \sin\theta \, e^{-\frac{ipr\cos\theta}{\hbar}}$$

$$= 2\pi V_0 \int_0^\infty dr \, e^{-\mu r} r \left[\frac{\hbar}{ipr} e^{-\frac{ipr}{\hbar}\cos\theta} \right]\Bigg|_0^\pi$$

$$= \frac{2\pi\hbar V_0}{ip} \int_0^\infty dr \, e^{-\mu r} \left\{ e^{\frac{ipr}{\hbar}} - e^{-\frac{ipr}{\hbar}} \right\} = \frac{2\pi\hbar V_0}{ip} \int_0^\infty dr \left\{ e^{r(-\mu + \frac{ip}{\hbar})} - e^{-r(\mu + \frac{ip}{\hbar})} \right\}$$

$p = \hbar k$

$$= \frac{2\pi\hbar V_0}{ip} \left[\frac{1}{-\mu + ik} e^{r(-\mu+ik)} + \frac{1}{\mu + ik} e^{-r(\mu+ik)} \right]\Bigg|_0^\infty$$

what about the e^{irk} and e^{-irk} from 0 to ∞? No value at ∞?

No value but $e^{-\mu r}$ already 0

$$= \frac{2\pi\hbar V_0}{ip} \left\{ \frac{1}{\mu - ik} - \frac{1}{\mu + ik} \right\} = \frac{2\pi\hbar V_0}{ip} \left\{ \frac{(\mu+ik) - (\mu - ik)}{(\mu-ik)(\mu+ik)} \right\}$$

$$= \frac{2\pi\hbar V_0}{ip} \frac{2ik}{\mu^2 + k^2} = \frac{4\pi V_0}{\mu^2 + k^2}$$

why $f^{(1)}$ negative? physical meaning?

As $f^{(1)}(\vec{k}, \vec{k}') = -\frac{m}{2\pi\hbar^2} \underbrace{\langle \vec{k}' | V | \vec{k} \rangle}_{\text{Fourier-Transform at } (\vec{k} - \vec{k}')}$, we get that

$$f^{(1)}(\vec{k}, \vec{k}') = -\frac{m}{2\pi\hbar^2} \frac{4\pi V_0}{\mu^2 + (\vec{k} - \vec{k}')^2}$$

$$= -\frac{2mV_0}{\hbar^2} \frac{1}{\mu^2 + q^2}, \text{ by introducing } q^2 := (\vec{k} - \vec{k}')^2$$

Same calculation as in real space (see lecture)?

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b) See next sheet

c) to be shown only at 2nd order

$$\mathrm{Im}\{ f(\vec{k}, \vec{k}) \} \overset{!}{=} \frac{k}{4\pi} \int d\Omega_{k'} \, | f(\vec{k}, \vec{k}') |^2$$

Why do only the $f^{(2)}$ and $|f^{(1)}|^2$ terms correspond? Why no connection between $f^{(1)}$ and $|f^{(2)}|^2$ for example?

$$\mathrm{Im}\{ f^{(2)}(\vec{k}, \vec{k}) \} = \mathrm{Im}\left\{ \frac{2m^2 V_0^2}{\hbar^4 \mu^2 (\mu - 2ik)} \right\} = \mathrm{Im}\left\{ \frac{2m^2 V_0^2 (\mu + 2ik)}{\hbar^4 \mu^2 (\mu^2 + 4k^2)} \right\}$$

$$= \mathrm{Im}\left\{ \frac{2m^2 V_0^2 \mu + i(4m^2 V_0^2 k)}{\hbar^4 \mu^2 (\mu^2 + 4k^2)} \right\}$$

$$= \frac{4m^2 V_0^2 k}{\hbar^4 \mu^2 (\mu^2 + 4k^2)} \qquad (*)$$

Inhomogenous

no order and really [illegible]

And

$$\frac{k}{4\pi}\int d\Omega_{k'} \left|f^{(1)}(\vec{k},\vec{k}')\right|^2 = \frac{k}{4\pi}\int d\Omega_{k'} \frac{4m^2V_0^2}{\hbar^4}\frac{1}{(\mu^2+q^2)^2}$$

$$\xrightarrow{q=\vec{k}-\vec{k}'} \frac{k}{4\pi}\frac{4m^2V_0^2}{\hbar^4}\int d\Omega_{k'}\frac{1}{(\mu^2+(\vec{k}-\vec{k}')^2)^2}$$

orientation of coordinate system such that θ' is angle between $\vec{k}$ and $\vec{k}'$ and $|\vec{k}|=k$, $|\vec{k}'|=k'$

$$= \frac{km^2V_0^2}{\hbar^4\pi}\int d\Omega_{k'}\frac{1}{(\mu^2+k^2+k'^2-2kk'\cos\theta')^2} = \frac{km^2V_0^2}{\hbar^4\pi}\int_0^{2\pi}d\varphi'\int_0^{\pi}d\theta'\frac{1\cdot\sin\theta'}{(\mu^2+k^2+k'^2-2kk'\cos\theta')^2}$$

why can you just set θ as the angle between k, k'?

$\boxed{\text{align the system}}$

no φ dependence

$$= \frac{2km^2V_0^2}{\hbar^4}\left[-\frac{1}{2kk'}\frac{1}{\mu^2+k^2+k'^2-2kk'\cos\theta'}\right]\Bigg|_0^{\pi}$$

$$= \frac{2km^2V_0^2}{2\hbar^4kk'}\left\{\frac{1}{\mu^2+k^2+k'^2-2kk'}-\frac{1}{\mu^2+k^2+k'^2+2kk'}\right\}$$

$$= \frac{m^2V_0^2}{\hbar^4k'}\left\{\frac{[\mu^2+(k+k')^2]-[\mu^2+(k-k')^2]}{[\mu^2+(k-k')^2][\mu^2+(k+k')^2]}\right\}$$

$$= \frac{m^2V_0^2}{\hbar^4k'}\left\{\frac{4kk'}{[\mu^2+(k-k')^2][\mu^2+(k+k')^2]}\right\}$$

$$\xrightarrow{k=k'} = \frac{4m^2V_0^2k}{\hbar^4\mu^2}\left\{\frac{1}{\mu^2+4k^2}\right\}$$

Comparing this with (*) yields that the optical theorem holds at 2^{nd} order.

why do we have to set $k=k'$?

$\boxed{q^2=2k^2(1-\cos\theta)}$

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d) To compare the magnitudes of the 1^{st} and 2^{nd} order Born terms for forward scattering (Yukawa potential), we calculate:

$$\left|\frac{f^{(2)}(\vec{k},\vec{k})}{f^{(1)}(\vec{k},\vec{k})}\right| = \left|\frac{2m^2V_0^2\cdot\mu^2\hbar^2}{\hbar^4\mu^2(\mu-2ik)\cdot 2mV_0}\right| = \left|\frac{mV_0}{\hbar^2(\mu-2ik)}\right|$$

$$= \left|\frac{mV_0(\mu+2ik)}{\hbar^2(\mu^2+4k^2)}\right| = \frac{m|V_0|}{\hbar^2}\frac{\sqrt{\mu^2+4k^2}}{\mu^2+4k^2}$$

$m, V_0, \mu, k > 0$

$\boxed{V_0<0 \text{ possible}}$

$$= \frac{m|V_0|}{\hbar^2}\frac{1}{\sqrt{\mu^2+4k^2}} \xrightarrow{k\ll\mu} \frac{mV_0}{\hbar^2\mu} \quad (1) = R$$

You can simplify your life & state:

$R \ll 1$ (for the different energy ranges)

$$\xrightarrow{k\gg\mu} \frac{mV_0}{2\hbar k} \quad (2) = R'$$

$\mu \hat{=}$ mass?

$\boxed{\text{eff. range}}$

- For small energies, (1) needs to be small, so either small mass or Potential V_0, or huge μ
- For high energies, (2) has to be small, which means big k (is fulfilled anyways as we are talking about high energies?) or small V_0, m again.

is $k\gg1$ if $k\gg\mu$?

$\boxed{\text{No}}$

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b) $f^{(2)}(\vec k, \vec k) = -\frac{m}{2\pi\hbar^2}\langle \vec k|\hat V \frac{1}{E-\hat H_0+i\epsilon}\hat V|\vec k\rangle$ ← no $\vec k'$

$$= -\frac{1}{(2\pi)^6}\frac{m}{2\pi\hbar^2}\iint d^3k'\, d^3k''\, \langle \vec k|\hat V|\vec k'\rangle\langle \vec k'|\frac{1}{E-\hat H_0+i\epsilon}|\vec k''\rangle\langle\vec k''|\hat V|\vec k\rangle$$

$\langle\vec k|V|\vec k'\rangle$ from a)
$$= -\frac{1}{(2\pi)^6}\frac{m}{2\pi\hbar^2}\int d^3k'\int d^3k'' \frac{4\pi V_0}{\mu^2+(\vec k'-\vec k)^2}\cdot\frac{4\pi V_0}{\mu^2+(\vec k-\vec k'')^2}\langle\vec k'|\frac{1}{E-\hat H_0+i\epsilon}|\vec k''\rangle$$

Also diagonal in energy space?

$\hat H_0$ diagonal in k space
$$= -\frac{m}{(2\pi)^7\hbar^2}\cdot\frac{16\pi^2V_0^2}{1}\int d^3k'\int d^3k''\frac{1}{\mu^2+(\vec k'-\vec k)^2}\cdot\frac{1}{\mu^2+(\vec k-\vec k'')^2}\,\frac{1}{E-E_0'+i\epsilon}\langle\vec k'|\vec k''\rangle$$

$\langle\vec k'|\vec k''\rangle = (2\pi)^3\delta(\vec k'-\vec k'')$
$$= -\frac{16m\pi^2V_0^2}{(2\pi)^4\hbar^2}\int d^3k'\frac{1}{[\mu^2+(\vec k-\vec k')^2]^2}\,\frac{1}{E-\frac{\hbar^2k'^2}{2m}+i\epsilon}$$

Is $E=\frac{\hbar^2k^2}{2m}$ or $E=\frac{\hbar^2\ell^2}{2m}$ for some ℓ?

$E=\frac{\hbar^2k^2}{2m}$, $\epsilon'=\frac{2m}{\hbar^2}\epsilon$
$$= -\frac{32m^2\pi^2V_0^2}{(2\pi)^4\hbar^4}\int d^3k'\frac{1}{[\mu^2+(\vec k-\vec k')^2]^2}\,\frac{1}{k^2-k'^2+i\epsilon'}$$

polar coordinates
$$= -\frac{32m^2\pi^2V_0^2}{(2\pi)^4\hbar^4}\int_0^\infty dk'\frac{k'^2}{k^2-k'^2+i\epsilon'}\int_0^{2\pi}d\varphi\int_0^\pi d\theta\,\sin\theta\,\frac{1}{(\mu^2+k^2+k'^2-2kk'\cos\theta)^2}$$

$$= -\frac{64m^2\pi^3V_0^2}{(2\pi)^4\hbar^4}\int_0^\infty dk'\frac{k'^2}{k^2-k'^2+i\epsilon'}\left[\frac{1}{\mu^2+k^2+k'^2-2kk'\cos\theta}\cdot\left(-\frac{1}{2kk'}\right)\right]_0^\pi$$

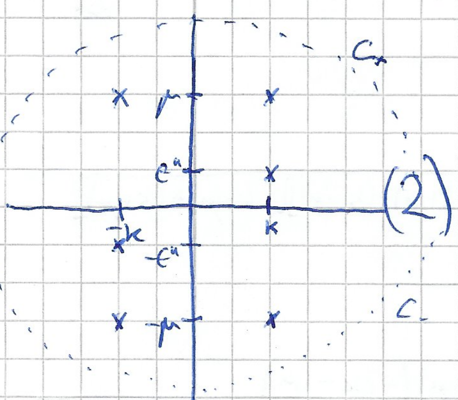
$$= -\frac{64m^2\pi^3V_0^2}{(2\pi)^4\hbar^4}\int_0^\infty dk'\frac{k'^2\left(\frac{1}{2kk'}\right)}{k^2-k'^2+i\epsilon'}\left\{\frac{1}{\mu^2+(k-k')^2}-\frac{1}{\mu^2+(k+k')^2}\right\}$$

$$= -\frac{32m^2\pi^3V_0^2}{(2\pi)^4\hbar^4k}\int_0^\infty dk'\frac{k'}{k^2-k'^2+i\epsilon'}\left\{\frac{[\mu^2+(k+k')^2]-[\mu^2+(k-k')^2]}{[\mu^2+(k-k')^2][\mu^2+(k+k')^2]}\right\}$$

$$= -\frac{32m^2\pi^3V_0^2}{\hbar^4k(2\pi)^4}\int dk'\frac{k'}{k^2-k'^2+i\epsilon'}\,\frac{4kk'}{[\mu^2+(k-k')^2][\mu^2+(k+k')^2]}$$

$$= \frac{128m^2\pi^3V_0^2}{\hbar^4(2\pi)^4}\int_0^\infty dk'\frac{k'^2}{k'^2-k^2-i\epsilon'}\,\frac{1}{[\mu^2+(k-k')^2][\mu^2+(k+k')^2]}$$

(1) this is symmetric under $k'\mapsto -k'$



"Interlude":

• $[k'-(k+i\epsilon'')][k'+(k+i\epsilon'')]$

$= k'^2-(k+i\epsilon'')^2 = k'^2-k^2+\underbrace{\epsilon''^2}_{\to 0 \text{ faster than } \epsilon''}\underbrace{-2ik\epsilon''}_{\hat{=}\, i\epsilon' \text{ with } \epsilon''=\frac{\epsilon'}{2k}}$

Nice!

• $[k'-(k+i\mu)][k'-(k-i\mu)]$

$= k'^2-k'(k+i\mu)-k'(k-i\mu)+(k+i\mu)(k-i\mu)$

$= k'^2-2kk'+k^2+\mu^2 = \mu^2+(k-k')^2$

analogue

• $[k'+(k+i\mu)][k'+(k-i\mu)] = \mu^2+(k+k')^2$

(3) $$= \frac{128m^2\pi^3V_0^2}{\hbar^4(2\pi)^4}\underbrace{\int_0^\infty dk'\frac{k'^2}{[k'-(k+i\epsilon'')][k'+(k+i\epsilon'')][k'-(k+i\mu)][k'-(k-i\mu)][k'+(k+i\mu)][k'+(k-i\mu)]}}_{(4)}$$

We want to calculate this using (1) and the residue theorem with (2) we get:

$$(4) = \frac{1}{2}\int_{-\infty}^{\infty} dk' \frac{k'^2}{[k'-(k+i\epsilon'')][k'+(k+i\epsilon'')][k'-(k+i\mu)][k'-(k-i\mu)][k'+(k+i\mu)][k'+(k-i\mu)]}$$

$$= \frac{1}{2}\int_{C_+} dk' \underbrace{\frac{k'^2}{[k'-(k+i\epsilon'')][k'+(k+i\epsilon'')][k'-(k+i\mu)][k'-(k-i\mu)][k'+(k+i\mu)][k'+(k-i\mu)]}}_{g(k'):=}$$

Where C_+ is the upper arc from $-\infty$ to ∞, where this doesn't change the value of the integral, as for $\operatorname{Im}\{C_+\} > 0$, the value for k' will always be ∞ on the arc and the denominator in the integral is of order 6 in k' while the numerator has order 2.

$$= \frac{1}{2}\cdot 2\pi i \left\{ \operatorname{Res}(g, k+i\epsilon'') + \operatorname{Res}(g, k+i\mu) + \operatorname{Res}(g, -k+i\mu) \right\}$$ [5]

So we reduced the problem to calculating 3 residues of order 1.

$$\operatorname{Res}(g, k+i\epsilon'') = [k'-(k+i\epsilon'')]\, g(k')\Big|_{k'=k+i\epsilon''}$$

$$= \frac{(k+i\epsilon'')^2}{[(k+i\epsilon'')+(k+i\epsilon'')][(k+i\epsilon'')-(k+i\mu)][(k+i\epsilon'')-(k-i\mu)][(k+i\epsilon'')+(k+i\mu)][(k+i\epsilon'')+(k-i\mu)]}$$

$$= \frac{(k+i\epsilon'')^2}{2[k+i\epsilon''][i\epsilon''-i\mu][i\epsilon''+i\mu][2k+i(\epsilon''+\mu)][2k+i(\epsilon''-\mu)]}$$

Where exactly can we do the limit $\epsilon \to 0$?

$\epsilon \to 0 \Leftrightarrow \epsilon' \to 0 \Leftrightarrow \epsilon'' \to 0$

$$= \frac{k}{2[-i\mu][i\mu][2k+i\mu][2k-i\mu]}$$

$$= \frac{k}{2\mu^2(4k^2+\mu^2)}$$

$$\operatorname{Res}(g, k+i\mu) = \frac{(k+i\mu)^2}{[(k+i\mu)-(k+i\epsilon'')][(k+i\mu)+(k+i\epsilon'')][(k+i\mu)-(k-i\mu)][(k+i\mu)+(k+i\mu)][(k+i\mu)+(k-i\mu)]}$$

$$= \frac{(k+i\mu)^2}{[i(\mu-\epsilon'')][2k+i(\mu+\epsilon'')][2i\mu]\,2[k+i\mu][2k]}$$

„ignore identical $i\epsilon$"? What does it mean in solution for P3?

$\epsilon'' \to 0$

$$= \frac{k+i\mu}{(i\mu)(2k+i\mu)(2i\mu)4k} = -\frac{(k+i\mu)(2k-i\mu)}{8\mu^2 k(4k^2+\mu^2)}$$

$$= -\frac{2k^2+\mu^2+ki\mu}{8\mu^2 k(4k^2+\mu^2)}$$

$$\operatorname{Res}(g, -k+i\mu) = \frac{(-k+i\mu)^2}{[(-k+i\mu)-(k+i\epsilon'')][(-k+i\mu)+(k+i\epsilon'')][(-k+i\mu)-(k+i\mu)][(-k+i\mu)-(k-i\mu)][(-k+i\mu)+(k+i\mu)]}$$

$$= \frac{(-k+i\mu)^2}{[-2k+i(\mu-\epsilon'')][i(\mu+\epsilon'')][-2k][-2k+2i\mu][2i\mu]}$$

$\epsilon'' \to 0$

$$= \frac{(-k+i\mu)}{2[-2k+i\mu](i\mu)(-2k)(2i\mu)} = +\frac{(-k+i\mu)(-2k-i\mu)}{8\mu^2 k(4k^2+\mu^2)}$$

$$= \frac{+2k^2 - i\mu k + \mu^2}{8\mu^2 k(4k^2+\mu^2)}$$

Putting this together, (5) is equal to

$$\pi i \left\{ \frac{k}{2\mu^2(4k^2+\mu^2)} - \frac{2k^2+\mu^2+ik\mu}{8\mu^2 k(4k^2+\mu^2)} + \frac{2k^2-i\mu k+\mu^2}{8\mu^2 k(4k^2+\mu^2)} \right\}$$

$$= i\pi \left\{ \frac{4k^2-2k^2-\mu^2-ik\mu+2k^2-i\mu k+\mu^2}{8\mu^2 k(4k^2+\mu^2)} \right\}$$

$$= i\pi \left\{ \frac{4k^2-2i\mu k}{8\mu^2 k(4k^2+\mu^2)} \right\} = i\pi \left\{ \frac{2k-i\mu}{4\mu^2(4k^2+\mu^2)} \right\}$$

$$= \frac{(2ik+\mu)\pi}{4\mu^2(4k^2+\mu^2)} = \frac{(\mu^2+4k^2)\pi}{4\mu^2(4k^2+\mu^2)(\mu-2ik)}$$

$$= \frac{\pi}{4\mu^2(\mu-2ik)}$$

So for (4), we get: $\frac{\pi}{4\mu^2(\mu-2ik)}$

So (3) becomes:

$$(3) = \frac{128 m^2\pi^3 V_0^2}{\hbar^4 (2\pi)^4} \frac{\pi}{4\mu^2(\mu-2ik)}$$

$$= \frac{32 m^2 \pi^4 V_0^2}{\hbar^4 (2\pi)^4 \mu^2(\mu-2ik)} = \frac{2m^2 V_0^2}{\hbar^4 \mu^2(\mu-2ik)}$$

$$= f^{(2)}(\vec{k},\vec{k})$$

Is it so much more difficult to calculate $f^{(2)}(\vec{k},\vec{k}')$ $\uparrow \vec{k}'$?

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