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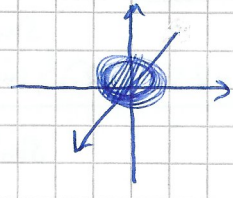
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H4) $V(r) = \begin{cases} \infty & r \leq R \\ 0 & r > R \end{cases}$



Does very low scattering energies imply $kR \ll 1$? which l change?

$\tan \delta_l(k) = \frac{j_l(kR)}{n_l(kR)}$
 $f_l(k) = \frac{e^{2i\delta_l(k)} - 1}{2ik} = \frac{1}{k \cot \delta_l(k) - ik}$

smaller atoms \rightarrow higher energies

a) For now: $kR \ll 1$ (low energy behaviour)

$f_0(k) = \frac{e^{2i\delta_0(k)} - 1}{2ik} = \frac{e^{-2ikR} - 1}{2ik}$ already small for $kR \ll 1$

as $\tan \delta_0(k) = \frac{j_0(kR)}{n_0(kR)} = -\frac{\sin(kR)}{\cos(kR)} = -\tan(kR) = \tan(-kR)$

$\Rightarrow \delta_0(k) = -kR \xrightarrow{k \text{ small}} 0$

and for $kR \ll 1$ the following holds (See Appendix - (19)):

$\tan \delta_l(k) = \frac{j_l(kR)}{n_l(kR)} \approx -\frac{(kR)^e}{(2e+1)!!} \frac{(kR)^{e+1}}{(2e-1)!!} = -\frac{(kR)^{2e+1}}{(2e+1)!!(2e-1)!!} \ll 1$
 for any l bigger than 1 and $kR \ll 1$
 $(\tan \delta_{e+1}(k) \ll \tan \delta_l(k))$

$\Rightarrow \delta_l(k) \ll 1$ as $\tan(x)$ bijective

$\Rightarrow f_l(k) \approx \frac{e^0 - 1}{2ik} = 0$ for $l \geq 1$ (note: also holds for $l=0$, stronger approximation)

What for are the effective range expansions for different l than $l=0$?

b) $\psi(k) := k \cot \delta_0(k) = \frac{k \cos \delta_0(k)}{\sin \delta_0(k)} = \frac{k \cos(-kR)}{\sin(-kR)} = -\frac{k \cos(kR)}{\sin(kR)}$

$\psi'(k) = -\frac{\{\cos(kR) - k \sin(kR)R\} \sin(kR) - \cos(kR) \cdot R \cdot k \cos(kR)}{\sin^2(kR)}$

$= -\frac{\cos(kR) \sin(kR) - kR \{\sin^2(kR) + \cos^2(kR)\}}{\sin^2(kR)}$

$= -\frac{\sin(kR) \cos(kR) - kR}{\sin^2(kR)}$

I would not say this since the theory is trivial if you set the first term in your perturbation theory ($l=0$) to zero!

What if you write this as $\frac{\cos(kR)}{\sin(kR)} \frac{kR}{\sin^2(kR)}$
 \rightarrow can't use l'Hopital? (see next page)
 $\frac{\cos(kR)}{\sin(kR)} = \frac{1}{0}$
 \uparrow kR small

$$\begin{aligned}
 y''(k) &= - \frac{\{-R + R \cos^2(kR) - R \sin^2(kR)\} \sin^4(kR) - 2 \sin(kR) R \sin(kR) \{ \sin^2(kR) - 2 \sin(kR) \cdot R \cdot \cos(kR) \} \cos(kR) \sin(kR) - kR^2}{\sin^4(kR)} \\
 &= - \frac{\{-R + R \cos^2(kR) - R \sin^2(kR)\} \sin^4(kR) - 2R \sin^2(kR) \cos^2(kR) + 2R^2 k \sin(kR) \cos(kR)}{\sin^4(kR)} \\
 &= - \frac{\{-R \sin^2(kR) - R \cos^2(kR) \sin^2(kR) - R \sin^4(kR) + 2R^2 k \sin(kR) \cos(kR)\}}{\sin^4(kR)} \\
 &= - \frac{-2R \sin^2(kR) + 2R^2 k \sin(kR) \cos(kR)}{\sin^4(kR)} = \frac{2R \sin(kR) - 2R^2 k \cos(kR)}{\sin^3(kR)}
 \end{aligned}$$

We want to calculate the Taylor series up to order k^2 .

$$T_y(k, 0) = y(0) + y'(0)k + \frac{y''(0)}{2}k^2 + O(k^3)$$

Recall $y(k) = \frac{-k \cos(kR)}{\sin(kR)}$ We need L'Hospital to calculate $y(0)$.

$$\lim_{k \rightarrow 0} y(k) = \lim_{k \rightarrow 0} \frac{-\cos(kR) + kR \sin(kR)}{\cos(kR) \cdot R} = -\frac{1}{R}$$

Recall $y'(k) = -\frac{\sin(kR) \cos(kR) - kR}{\sin^2(kR)}$ Again $y'(0) = \frac{0}{0}$ if calculated primitively

$$\lim_{k \rightarrow 0} y'(k) = \lim_{k \rightarrow 0} -\frac{\cos^2(kR) \cdot R - \sin^2(kR) \cdot R - R}{2 \sin(kR) \cos(kR) R} = \lim_{k \rightarrow 0} \frac{(-R)}{(R)} \frac{\cos^2(kR) - \sin^2(kR) - 1}{2 \sin(kR) \cos(kR)}$$

$$= \lim_{k \rightarrow 0} \frac{(-1)}{2} \frac{-2 \cos(kR) \sin(kR) \cdot R - 2 \sin(kR) \cos(kR) \cdot R}{\cos^2(kR) \cdot R - \sin^2(kR) \cdot R} = \lim_{k \rightarrow 0} \left(-\frac{1}{2}\right) \frac{-4 \sin(kR) \cos(kR)}{\cos^2(kR) - \sin^2(kR)}$$

$$= \lim_{k \rightarrow 0} 2 \frac{\sin(kR) \cos(kR)}{\cos^2(kR) - \sin^2(kR)} = 0$$

Same for $y''(k)$ yields:

$$\lim_{k \rightarrow 0} y''(k) = \lim_{k \rightarrow 0} \frac{\{-\sin^2(kR) - 2k \sin(kR) \cos(kR)\} R^2 - R \{-2 \cos(kR) \sin^3(kR) - R + 2 \sin(kR) \cos^3(kR)\} - 4 \cos^2(kR) \cos(kR) \cdot R}{4 \sin^3(kR) \cos(kR) \cdot R}$$

$$\begin{aligned}
 &= \lim_{k \rightarrow 0} \left(\frac{1}{R}\right) \frac{\sin^2(kR) + 2kR \sin(kR) \cos(kR) + 2R^2 \sin^3(kR) \cos(kR) + 2R^2 \cos^3(kR) \sin(kR)}{4 \sin^3(kR) \cos(kR)} \\
 &\quad \dots = \frac{-2R^2 \sin(kR) \cos(kR) - 2R^3 \cos^2(kR) + 2R^3 k \sin^2(kR)}{4 \sin^3(kR) \cos(kR)}
 \end{aligned}$$

$$\begin{aligned}
 &= \lim_{k \rightarrow 0} \left(\frac{1}{R}\right) \frac{\sin^2(kR) + 2kR \sin(kR) \cos(kR) - 2R^3 k \cos^2(kR) + 2R^3 k \sin^2(kR)}{4 \sin^3(kR) \cos(kR)}
 \end{aligned}$$

We fucked up the 2nd derivative of $\psi(k)$, so again for the correct $\psi''(k)$ now:

L'Hospital yields:

$$\lim_{k \rightarrow 0} \psi''(k) = \lim_{k \rightarrow 0} \frac{2R^2 \cos(kR) - 2R^2 \{ \cos(kR) - k \sin(kR) \} R}{3 \sin^2(kR) \cos(kR) R}$$

$$= \lim_{k \rightarrow 0} \frac{2R^2 k \sin(kR)}{3 \sin^2(kR) \cos(kR)} = \lim_{k \rightarrow 0} \frac{2R^2 k}{3 \sin(kR) \cos(kR)}$$

$$= \lim_{k \rightarrow 0} \frac{2R^2}{3 \{ \cos^2(kR) R - \sin^2(kR) R \}} = \lim_{k \rightarrow 0} \frac{2R}{3 \{ \cos^2(kR) - \sin^2(kR) \}}$$

$$= \frac{2}{3} R$$

IS the k term always zero?

$\Rightarrow T\psi(k,0) = -\frac{1}{R} + \frac{1}{3} R k^2 + O(k^4)$

$\stackrel{\nabla}{=} -\frac{1}{a_0} + \frac{R}{2} k^2 + \dots$

$\Rightarrow a_0 = R \quad \& \quad r_0 = \frac{2}{3} R$

If we were more attentive, we would have seen the hint that

$$z \cot z = 1 - \frac{1}{3} z^2 - \frac{1}{45} z^4 - \dots$$

and this makes the b) a 2-linear...

There are those Bernoulli numbers

But coming from in Taylor expansion of $z \cot z$?

c) $\sigma_{\text{tot}} = \frac{4\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1) \sin^2 \delta_{\ell}(k) \tan \delta_{\ell}(k) \approx \left. \begin{aligned} & \frac{k \cot k}{k} \left\{ - \frac{(kR)^{2\ell+1}}{(2\ell+1)!(\ell!)!} \right\} \\ & \approx \delta_{\ell}(k) \approx \sin \delta_{\ell}(k) \end{aligned} \right\}$

Why can't we just use this? Yields a different result!
 $= \frac{4\pi}{k^2} \{ kR^2 + \frac{2}{3} kR^4 + \dots \}$

$$= \int dR |f_{\ell}(R)|^2 \approx 2 \cdot 2\pi \sum_{\ell=0}^{\infty} (2\ell+1) |f_{\ell}(k)|^2$$

$$\approx 4\pi |f_0(k)|^2$$

Taylor $f_0(k)$ at $k=0$

$\Rightarrow f_0(k) = \frac{1}{k \cot \delta_0(k) - ik} \stackrel{b)}{=} \frac{1}{-\frac{1}{R} + \frac{R}{3} k^2 - ik} = -R \frac{1}{1 - \frac{R^2}{3} k^2 + iRk}$

$\stackrel{\text{hint}}{\approx} -R \left\{ 1 - iRk + \left(-R^2 + \frac{R}{3} k^2 \right) \right\}$

So we get

$$\begin{aligned} \sigma_{\text{tot}} &\approx 4\pi |f_0(k)|^2 = 4\pi \left\{ R^2 \left(1 - \frac{2}{3} R^2 k^2 \right) - iRk \right\} \\ &= 4\pi R^2 \left\{ \left(1 - \frac{2}{3} R^2 k^2 \right)^2 + R^2 k^2 \right\} \\ &= 4\pi R^2 \left\{ 1 + \frac{4}{9} R^4 k^4 - \frac{4}{3} R^2 k^2 + R^2 k^2 \right\} \\ &= 4\pi R^2 \left\{ 1 - \frac{1}{3} (kR)^2 + \mathcal{O}(kR)^4 \right\} \quad \square \end{aligned}$$

d) Now: $kR \gg 1$ (High energy behaviour)

$$\sin^2 \delta_l(k) = \sin^2 \left(kR + \frac{\alpha_l}{2} \right) = \sin^2 \left(kR - \frac{\alpha_l}{2} \right)$$

because $\tan \delta_l(k) = \frac{j_l(kR)}{n_l(kR)} \underset{\substack{\text{high energies} \\ (\text{Asymptotic})}}{\approx} - \frac{\sin \left(kR - \frac{\alpha_l}{2} \right)}{\cos \left(kR - \frac{\alpha_l}{2} \right)} = -\tan \left(kR - \frac{\alpha_l}{2} \right) = \tan \left(-kR + \frac{\alpha_l}{2} \right)$

Why can we use $kR \gg 1$ for δ_l, δ_{l+1} etc. and do not have to calculate the exact $\delta_l(k)$?

$$\Rightarrow \delta_l(k) = -kR + \frac{\alpha_l}{2} \quad \square$$

$$\tan \delta_l = \frac{j_l(kR)}{n_l(kR)}$$

e) Using (d): $\sin^2 \delta_l(k) + \sin^2 \delta_{l+1}(k) \approx \sin^2 \left(kR - \frac{\alpha_l}{2} \right) + \sin^2 \left(kR - \frac{\alpha_{l+1}}{2} \right)$

$$\begin{aligned} &= \sin^2 \left(kR - \frac{\alpha_l}{2} \right) + \sin^2 \left(kR - \frac{\alpha_l}{2} - \frac{\pi}{2} \right) \\ &= \sin^2 \left(kR - \frac{\alpha_l}{2} \right) + \left(-\cos \left(kR - \frac{\alpha_l}{2} \right) \right)^2 \\ &= \sin^2 \left(kR - \frac{\alpha_l}{2} \right) + \cos^2 \left(kR - \frac{\alpha_l}{2} \right) = 1 \end{aligned}$$

Why $\alpha_{l+1} = \alpha_l + \pi$?

$$\sigma_{\text{high}} = \frac{4\pi}{k^2} \sum_{l=0}^{l_{\text{max}}} (2l+1) \sin^2 \delta_l(k) = \frac{4\pi}{k^2} \left\{ \sum_{l=0}^{l_{\text{max}}} (2l+1) \sin^2 \delta_l(k) + \sum_{l=0}^{l_{\text{max}}} l \sin^2 \delta_l(k) \right\}$$

$$\approx \frac{4\pi}{k^2} \left\{ \sum_{l=0}^{l_{\text{max}}} (2l+1) \sin^2 \delta_l(k) + \sum_{l=0}^{l_{\text{max}}-1} (2l+1) \sin^2 \delta_{l+1}(k) \right\}$$

This is incorrect! You should say $\approx \frac{4\pi}{k^2} \sum_{l=0}^{l_{\text{max}}} (2l+1) \sin^2 \delta_l(k)$

(e) and index shift $l \rightarrow 0$

$$\approx \frac{4\pi}{k^2} \left\{ (2l_{\text{max}}+1) \sin^2 \delta_{l_{\text{max}}}(k) + \sum_{l=0}^{l_{\text{max}}-1} (2l+1) \right\}$$

For $kR \gg 1$, $\alpha_l(kR) + \pi(kR)^2 \approx \beta(kR)^2$

$$\begin{aligned} &= \frac{4\pi}{k^2} \left\{ (2l_{\text{max}}+1) \sin^2 \delta_{l_{\text{max}}}(k) + \frac{l_{\text{max}} + \frac{l_{\text{max}}(l_{\text{max}}-1)}{2}}{2} \right\} \\ &\approx \frac{4\pi}{k^2} \left\{ \frac{(kR)^2}{2} + \frac{4\pi}{k^2} \mathcal{O}(kR) \right\} \\ &\approx 2\pi R^2 \end{aligned}$$

\rightarrow for k large

$\frac{l_{\text{max}}(l_{\text{max}}+1)}{2} \approx \frac{l_{\text{max}}^2}{2}$ for large l_{max}

+) $\sigma_{\text{cl}} = \pi R^2$, area of possible interaction given by classical cross-section.

Tutorial $\sigma_{\text{high}} \approx 4\pi R^2$

b) $\sigma_{\text{cl}} < \sigma_{\text{high}} < \sigma_{\text{low}}$. In classical theory, there is only an interaction if the particle hits the sphere (no wave character) if the energy is low (particle lower) the chance of interaction is higher.