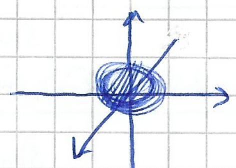
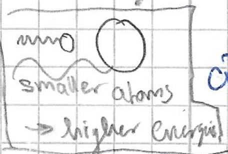
Disclaimer

09.11.2016 Advanced Quantum Theory Exercise 3 Marvin Zanke

H4) $V(r) = \begin{cases} \infty & , r \leq R \\ 0 & , r > R \end{cases}$



Does very low scattering energies k imply $kR \ll 1$? what is R large?



$$\tan \delta_\ell(k) = \frac{j_\ell(kR)}{n_\ell(kR)}$$

$$f_\ell(k) = \frac{e^{2i\delta_\ell(k)} - 1}{2ik} = \frac{1}{k\cot\delta_\ell(k) - ik}$$

a) For now: $kR \ll 1$; (low energy behaviour)

$$f_0(k) = \frac{e^{2i\delta_0(k)} - 1}{2ik} = \frac{e^{-2ikR} - 1}{2ik}$$ already small for $kR \ll 1$

as $\tan\delta_0(k) = \frac{j_0(kR)}{n_0(kR)} = -\frac{\sin(kR)}{\cos(kR)} = -\tan(kR) = \tan(-kR)$

$$\Leftrightarrow \delta_0(k) = -kR \xrightarrow{k \text{ small}} 0$$

and for $kR \ll 1$ the following holds (see Appendix (19)):

$$\tan\delta_\ell(k) = \frac{j_\ell(kR)}{n_\ell(kR)} \overset{kR \ll 1}{\approx} -\frac{(kR)^\ell}{(2\ell+1)!!} \frac{(kR)^{\ell+1}}{(2\ell-1)!!} = -\frac{(kR)^{2\ell+1}}{(2\ell+1)!!(2\ell-1)!!} \ll 1$$

for any ℓ bigger than 1 and $kR \ll 1$

$(\tan\delta_{\ell+1}(k) \ll \tan\delta_\ell(k))$

$\rightsquigarrow \delta_\ell(k) \ll 1$ as $\tan(x)$ bijective

$\rightsquigarrow f_\ell(k) \approx \frac{e^0 - 1}{2ik} = 0$ for $\ell \geq 1$ (note: also holds for $\ell = 0$, stronger approximation)

What for are the effective range expansions for different ℓ than $\ell = 0$?

b) $$\psi(k) := k\cot\delta_0(k) = \frac{k\cos\delta_0(k)}{\sin\delta_0(k)} = \frac{k\cos(-kR)}{\sin(-kR)} = -\frac{k\cos(kR)}{\sin(kR)}$$

$$\psi'(k) = -\frac{\{\cos(kR) - k\sin(kR)R\}\sin(kR) - \cos(kR)\cdot R\cdot k\cos(kR)}{\sin^2(kR)}$$

$$= -\frac{\cos(kR)\sin(kR) - kR\{\sin^2(kR) + \cos^2(kR)\}}{\sin^2(kR)}$$

$$= -\frac{\sin(kR)\cos(kR) - kR}{\sin^2(kR)}$$

I would not say this since the theory is trivial if you set the first term in your perturbation theory ($\ell = 0$) to zero!

what if you write this as $-\left\{\frac{\cos(kR)}{\sin(kR)} - \frac{kR}{\sin^2(kR)}\right\}$

→ Can't use l'hopital? (see next page)

$\frac{\cos(kR)}{\sin(kR)} = \frac{1}{0}$ ↑ kR small

$$\psi''(k) = -\frac{\{-R + \cos(kR)\cdot R\cos(kR) - \sin(kR)\, R\sin(kR)\}\{\sin^2(kR)\} - 2\sin(kR)\cdot R\cdot\cos(kR)\{\cos(kR)\sin(kR) - kR\}}{\sin^4(kR)}$$

$$= -\frac{\{-R + R\cos^2(kR) - R\sin^2(kR)\}\{\sin^2(kR)\} - 2R\sin^2(kR)\cos^2(kR) + 2R^2 k\sin(kR)\cos(kR)}{\sin^4(kR)}$$

$$= -\frac{\{-R\sin^2(kR) - R\cos^2(kR)\sin^2(kR) - R\sin^4(kR) + 2R^2 k\sin(kR)\cos(kR)\}}{\sin^4(kR)}$$

$$= -\frac{-2R\sin^2(kR) + 2R^2 k\sin(kR)\cos(kR)}{\sin^4(kR)} = \frac{2R\sin(kR) - 2R^2 k\cos(kR)}{\sin^3(kR)}$$

We want to calculate the Taylor series up to order k^2:

$$T\psi(k,0) = \psi(0) + \psi'(0)k + \frac{\psi''(0)}{2}k^2 + O(k^3)$$

Recall $\psi(k) = \frac{-k\cos(kR)}{\sin(kR)}$ We need L'hospital to calculate $\psi(0)$:

$$\lim_{k\to 0}\psi(k) = \lim_{k\to 0}\frac{-\cos(kR) + kR\sin(kR)}{\cos(kR)\cdot R} = -\frac{1}{R}$$

Recall $\psi'(k) = -\frac{\sin(kR)\cos(kR) - kR}{\sin^2(kR)}$ Again "$\psi'(0) = \frac{0}{0}$" if calculated primitively

$$\lim_{k\to 0}\psi'(k) = \lim_{k\to 0} -\frac{\cos^2(kR)\cdot R - \sin^2(kR)\cdot R - R}{2\sin(kR)\cos(kR)R} = \lim_{k\to 0}\frac{(-R)}{(R)}\frac{\cos^2(kR) - \sin^2(kR) - 1}{2\sin(kR)\cos(kR)}$$

$$= \lim_{k\to 0}\frac{(-1)}{2}\frac{-2\cos(kR)\sin(kR)\cdot R - 2\sin(kR)\cos(kR)\cdot R}{\cos^2(kR)\cdot R - \sin^2(kR)\cdot R} = \lim_{k\to 0}\left(-\frac{1}{2}\right)\frac{-4\sin(kR)\cos(kR)}{\cos^2(kR) - \sin^2(kR)}$$

$$= \lim_{k\to 0} 2\frac{\sin(kR)\cos(kR)}{\cos^2(kR) - \sin^2(kR)} = 0$$

~~Same for $\psi''(k)$ yields:~~

$$\lim_{k\to 0}\psi''(k) = \left[\lim_{k\to 0} -\frac{\{-\sin^2(kR) - 2k\sin(kR)\cos(kR)\, R\} - R\{-2\cos(kR)\sin^3(kR)\cdot R + 2\sin(kR)\cos^3(kR)R\} - 4R\sin^2(kR)\cos(kR)\cdot R \;\dots\; + 2R^2\{\sin(kR)\cos(kR) + k\cos^2(kR)R - k\sin^2(kR)R\}}{4\sin^3(kR)\cos(kR)\cdot R}\right]$$

$$= \lim_{k\to 0}\left(\frac{1}{R}\right)\left[\frac{\sin^2(kR) + 2kR\sin(kR)\cos(kR) + \overbrace{2R^2\sin^3(kR)\cos(kR) + 2R^2\cos^3(kR)\sin(kR)}^{2R^2\sin(kR)\cos(kR)} \;\dots\; - 2R^2\sin(kR)\cos(kR) - 2R^3 k\cos^2(kR) + 2R^3 k\sin^2(kR)}{4\sin^3(kR)\cos(kR)}\right]$$

$$= \lim_{k\to 0}\left(\frac{1}{R}\right)\frac{\sin^2(kR) + 2kR\sin(kR)\cos(kR) - 2R^3 k\cos^2(kR) + 2R^3 k\sin^2(kR)}{4\sin^3(kR)\cos(kR)}$$

We fucked up the 2^{nd} derivative of $\varphi(k)$, so again for the correct $\varphi''(k)$ now:

L'hospital yields:

$$\lim_{k\to 0} \varphi''(k) = \lim_{k\to 0} \frac{2R^2\cos(kR) - 2R^2\{\cos(kR) - k\sin(kR)R\}}{3\sin^2(kR)\cos(kR)R}$$

$$= \lim_{k\to 0} \frac{2R^2 k\sin(kR)}{3\sin^2(kR)\cos(kR)} = \lim_{k\to 0} \frac{2R^2 k}{3\sin(kR)\cos(kR)}$$

$$= \lim_{k\to 0} \frac{2R^2}{3\{\cos^2(kR)R - \sin^2(kR)R\}} = \lim_{k\to 0} \frac{2R}{3\{\cos^2(kR) - \sin^2(kR)\}}$$

$$= \frac{2}{3}R$$

Is the k term always zero?

$$\Rightarrow T\varphi(k,0) = -\frac{1}{R} + \frac{1}{3}Rk^2 + \mathcal{O}(k^3)$$

$$\overset{!}{=} -\frac{1}{a_e} + \frac{r_e}{2}k^2 + \dots$$

$$\Rightarrow a_e = R \quad \& \quad r_e = \frac{2}{3}R$$

There are those Bernoulli numbers Bn coming from in Taylor expansion of $z\cot z$?

If we were more attentive, we would have seen the hint that

$$z\cot z = 1 - \frac{1}{3}z^2 - \frac{1}{45}z^4 - \dots$$

and this makes the b) a 2-liner...

c) $$\sigma_{tot} = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(l+1)\sin^2\delta_l(k) \qquad \tan\delta_l(k) \overset{k\ll 1}{\approx} \{-\frac{(kR)^{2l+1}}{(2l+1)!!(2l-1)!!}\}$$

$$\approx \delta_l(k) \approx \sin\delta_l(k)$$

Why can't we just use this? Yields a different result!

$$= -\frac{4\pi}{k^2}\{(kR)^2 + \frac{3}{9}(kR)^6 + \dots\}$$

$$= \int d\Omega\, |f_k(\theta)|^2 \approx 2\cdot 2\pi \sum_{l=0}^{\infty}(2l+1)|f_l(k)|^2$$

$$\approx 4\pi\, |f_0(k)|^2$$

Taylor $f_0(k)$ at $k=0$

$$\Rightarrow f_0(k) = \frac{1}{k\cot\delta_0(k) - ik} \overset{b)}{=} \frac{1}{-\frac{1}{R} + \frac{R}{3}k^2 - ik} = -R\frac{1}{1 - \frac{R^2}{3}k^2 + iRk}$$

$$\overset{(hint)}{\approx} -R\{1 - iRk + (-R^2 + \frac{R^2}{3})k^2\}$$

So we get

$$\sigma_{tot} \approx 4\pi |f_0(k)|^2 = 4\pi \left\{ R^2 \left| \left(1 - \tfrac{2}{3}R^2k^2\right) - iRk \right| \right\}$$
$$= 4\pi R^2 \left\{ \left(1 - \tfrac{2}{3}R^2k^2\right)^2 + R^2k^2 \right\}$$
$$= 4\pi R^2 \left\{ 1 + \tfrac{4}{9}R^4k^4 - \tfrac{4}{3}R^2k^2 + R^2k^2 \right\}$$
$$= 4\pi R^2 \left\{ 1 - \tfrac{1}{3}(kR)^2 + \mathcal{O}((kR)^4) \right\} \quad \boxed{\text{M}}$$

d) Now: $kR \gg 1$ (High energy behaviour)

$$\sin^2\delta_l(k) = \sin^2\left(\cancel{-kR} + \tfrac{l\pi}{2}\right) = \sin^2\left(kR - \tfrac{l\pi}{2}\right)$$

because $\tan\delta_l(k) = \dfrac{j_l(kR)}{n_l(kR)} \underset{\substack{\text{high}\\\text{energies}\\\text{(Appendix)}}}{\approx} - \dfrac{\sin\left(kR - \frac{l\pi}{2}\right)}{\cos\left(kR - \frac{l\pi}{2}\right)} = -\tan\left(kR - \tfrac{l\pi}{2}\right) = \tan\left(-kR + \tfrac{l\pi}{2}\right)$

Why can we use $kR \gg 1$ for δ_1, δ_2, etc. and do not have to calculate the exact $\delta_l(k)$?

$\tan\delta_l = \dfrac{j_l(kR)}{n_l(kR)}$

$$\Leftrightarrow \delta_l(k) = -kR + \frac{l\pi}{2} \quad \boxed{\text{M}}$$

e) Using (d): $\sin^2\delta_l(k) + \sin^2\delta_{l+1}(k) \approx \sin^2\left(kR - \tfrac{l\pi}{2}\right) + \sin^2\left(kR - \tfrac{(l+1)\pi}{2}\right)$

$$= \sin^2\left(kR - \tfrac{l\pi}{2}\right) + \sin^2\left(kR - \tfrac{l\pi}{2} - \tfrac{\pi}{2}\right)$$
$$= \sin^2\left(kR - \tfrac{l\pi}{2}\right) + \left(-\cos\left(kR - \tfrac{l\pi}{2}\right)\right)^2$$
$$= \sin^2\left(kR - \tfrac{l\pi}{2}\right) + \cos^2\left(kR - \tfrac{l\pi}{2}\right) = 1 \quad \boxed{\text{M}}$$

why $l_{max} = kR$?

$$\sigma_{high} = \frac{4\pi}{k^2}\sum_{l=0}^{l_{max}}(2l+1)\sin^2\delta_l(k) = \frac{4\pi}{k^2}\left\{\sum_{l=0}^{l_{max}}(l+1)\sin^2\delta_l(k) + \sum_{l=0}^{l_{max}} l\sin^2\delta_l(k)\right\}$$
$$= \frac{4\pi}{k^2}\left\{\sum_{l=0}^{l_{max}}(l+1)\sin^2\delta_l(k) + \sum_{l=0}^{l_{max}-1}(l+1)\sin^2\delta_{l+1}(k)\right\}$$

(e) and index shift $l=0 \to 0$

$$\approx \frac{4\pi}{k^2}\left\{(l_{max}+1)\sin^2\delta_{l_{max}}(k) + \sum_{l=0}^{l_{max}-1}(l+1)\right\}$$
$$= \frac{4\pi}{k^2}\left\{(l_{max}+1)\sin^2\delta_{l_{max}}(k) + \underbrace{l_{max} + \frac{l_{max}(l_{max}-1)}{2}}_{\frac{l_{max}(l_{max}+1)}{2} \approx \frac{l_{max}^2}{2} \text{ for large } l_{max}}\right\}$$

$0 \le \sin^2(x) \le 1$

$$\approx \frac{4\pi}{k^2}\frac{(kR)^2}{2} + \underbrace{\frac{4\pi}{k^2}\mathcal{O}(kR)}_{\to 0 \text{ for } k \text{ large}}$$
$$\approx 2\pi R^2$$

This is incorrect! You should say $\approx \frac{4\pi}{k^2}\sum_{l=0}^{l_{max}}(l+1)\{\sin^2\delta_l + \sin^2\delta_{l+1}\}$

For $kR \gg 1$, $\alpha(kR) + \beta(kR)^2 \approx \beta(kR)^2$

from Tutorial

f) $\sigma_{cl} = \pi R^2$, area of possible interaction given by „classical cross-section".

Tutorial $\sigma_{low} \approx 2\sigma_{high} \approx 4\sigma_{cl}$

g) $\sigma_{cl} < \sigma_{high} < \sigma_{low}$. In classical theory, there is only an interaction if the particle hit the sphere (→ lower chance). If the energy is low (particle slower) the chance of interaction is higher.