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22.11.2016 Advanced Quantum theory Exercise 4 Marvin Zanke

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Phys. meaning of reaction matrix R?

H5)
a)

$R := S - \mathbb{1}$

$R + R^\dagger = (S-\mathbb{1}) + (S-\mathbb{1})^\dagger = S + S^\dagger - 2\cdot\mathbb{1}$ (*)

S is a unitarity operator $SS^\dagger = \mathbb{1}$ because of prob. conservation?

$-R^\dagger R = -(S-\mathbb{1})^\dagger (S-\mathbb{1}) = -(S^\dagger - \mathbb{1})(S-\mathbb{1})$

$\overset{\mathbb{1}^\dagger = \mathbb{1}}{=} -(S^\dagger S - S - S^\dagger + \mathbb{1}) \underset{S^\dagger = S^{-1}}{=} -(2\cdot\mathbb{1} - S - S^\dagger) = S + S^\dagger - 2\cdot\mathbb{1} = (*)$ ■

b) We recall some equations we will need for this prove:

Why change position of $\vec{k}, \vec{k}'$ in f here?

Lecture: $\begin{cases} f(\vec{k},\vec{k}') = -\frac{m}{2\pi\hbar^2}\langle \vec{k}' | T | \vec{k}\rangle & (1) \\ f(\vec{k}',\vec{k})^* = -\frac{m}{2\pi\hbar^2}\langle \vec{k}' | T^\dagger | \vec{k}\rangle & (2) \end{cases}$

Sheet: $\begin{cases} \langle \vec{k}' | R | \vec{k}\rangle = -2\pi i\, \delta(E_{k'} - E_k) \langle \vec{k}' | T | \vec{k}\rangle & (3) \\ \langle \vec{k} | R^\dagger | \vec{k}'\rangle = 2\pi i\, \delta(E_{k'} - E_k) \langle \vec{k} | T^\dagger | \vec{k}'\rangle & (4) \end{cases}$

What for did they give us $\langle x | \vec{k}\rangle \hat{=} e^{i\vec{k}\vec{x}}$?

$$\begin{aligned} \langle \vec{k}' | R + R^\dagger | \vec{k}\rangle &= \langle \vec{k}' | -R^\dagger R | \vec{k}\rangle \\ &= \langle \vec{k}' | R | \vec{k}\rangle + \langle \vec{k}' | R^\dagger | \vec{k}\rangle \\ &\overset{(3),(4)}{=} -2\pi i\, \delta(E_{k'} - E_k)\langle \vec{k}' | T | \vec{k}\rangle \\ &\quad + 2\pi i\, \delta(E_{k'} - E_k)\langle \vec{k}' | T^\dagger | \vec{k}\rangle \\ &= -2\pi i\, \delta(E_{k'} - E_k)\{\langle \vec{k}' | T - T^\dagger | \vec{k}\rangle\} \end{aligned} \quad (5)$$

Furthermore:

$$\begin{aligned} f(\vec{k},\vec{k}') - f(\vec{k}',\vec{k})^* &\overset{(1),(2)}{=} -\frac{m}{2\pi\hbar^2}\{\langle \vec{k}' | T | \vec{k}\rangle - \langle \vec{k}' | T^\dagger | \vec{k}\rangle\} \\ &= -\frac{m}{2\pi\hbar^2}\{\langle \vec{k}' | T - T^\dagger | \vec{k}\rangle\} \end{aligned}$$

Can't I also introduce some $\int dE_q\, \delta(E_q - E_{k'})$ here? But $q = k'$ doesn't hold here?

$$\begin{aligned} \Rightarrow \delta(E_{k'} - E_k)\{f(\vec{k},\vec{k}') - f(\vec{k}',\vec{k})^*\} \\ &= -\frac{m}{2\pi\hbar^2}\delta(E_{k'} - E_k)\langle \vec{k}' | T - T^\dagger | \vec{k}\rangle \overset{(5)}{=} \frac{m}{4\pi^2\hbar^2 i}\langle \vec{k}' | -R^\dagger R | \vec{k}\rangle \\ &= \frac{im}{4\pi^2\hbar^2}\langle \vec{k}' | R^\dagger R | \vec{k}\rangle = \frac{im}{4\pi^2\hbar^2}\int \frac{d^3q}{(2\pi)^3}\langle \vec{k}' | R^\dagger | \vec{q}\rangle\langle \vec{q} | R | \vec{k}\rangle \\ &= \frac{im}{4\pi^2\hbar^2}\int \frac{d^3q}{(2\pi)^3}\{2\pi i\, \delta(E_{k'} - E_q)\langle \vec{k}' | T^\dagger | \vec{q}\rangle\} \times \{-2\pi i\, \delta(E_q - E_k)\langle \vec{q} | T | \vec{k}\rangle\} \\ &= \frac{im}{(2\pi)^3\hbar^2}\int d^3q\, \underbrace{\delta(E_{k'} - E_q)\,\delta(E_k - E_q)}_{\delta(E_{k'} - E_k)\,\delta(E_q - E_k)}\langle \vec{k}' | T^\dagger | \vec{q}\rangle\langle \vec{q} | T | \vec{k}\rangle \end{aligned}$$

$\delta(x-z)\delta(y-z) = \delta(x-y)\delta(x-z)$?

$$= \frac{im}{(2\pi)^3\hbar^2} \delta(E_k - E_{k'}) \int d^3q\, \delta(E_q - E_k) \underbrace{\langle \vec{k}' | T^\dagger | \vec{q} \rangle}_{\text{with } (2)\ = -\frac{2\pi\hbar^2}{m} f(\vec{k}',\vec{q})^*} \underbrace{\langle \vec{q} | T | \vec{k} \rangle}_{\text{with } (1)\ = -\frac{2\pi\hbar^2}{m} f(\vec{k},\vec{q})}$$

$$= \frac{i\hbar^2}{2\pi m} \delta(E_k - E_{k'}) \int d^3q\, \delta(E_q - E_k)\, f(\vec{k}',\vec{q})^* f(\vec{k},\vec{q})$$

Why does it hold without the δ's?

$$\Rightarrow \delta(E_k - E_{k'}) \left\{ f(\vec{k},\vec{k}') - f(\vec{k},\vec{k}) \right\} = \delta(E_k - E_{k'}) \left\{ \frac{i\hbar^2}{2\pi m} \int d^3q\, \delta(E_q - E_k) f(\vec{k}',\vec{q})^* f(\vec{k},\vec{q}) \right\}$$

Setting $\hbar = 1$ yields the result, as this equation holds for any $\vec{k}, \vec{k}'$.

c) $$\operatorname{Im} f(\vec{k},\vec{k}) = \frac{1}{2i} \left\{ f(\vec{k},\vec{k}) - f(\vec{k},\vec{k})^* \right\} = \frac{1}{4\pi m} \int d^3q\, \delta(E_q - E_k)\, f(\vec{k},\vec{q})^* f(\vec{k},\vec{q})$$

$$= \frac{1}{4\pi m} \int d^3q\, \delta(E_q - E_k)\, |f(\vec{k},\vec{q})|^2$$

$$= \frac{1}{4\pi m} \int d\Omega_q \int dq\, \delta(E_q - E_k)\, |f(\vec{k},\vec{q})|^2\, q^2$$

Jac. det.

$E_q = \frac{\hbar^2 q^2}{2m} \Leftrightarrow \frac{dE_q}{dq} = \frac{\hbar^2 q}{m}$

$$= \frac{1}{4\pi m} \int d\Omega_q \int dE_q\, \frac{m}{\hbar^2 q}\, q^2\, \delta(E_q - E_k)\, |f(\vec{k},\vec{q})|^2$$

Why not working if using $\delta(ax) = \frac{1}{a}\delta(x)$ here $\Rightarrow q^2 = k^2$ etc.

$$= \frac{1}{4\pi\hbar^2} \int d\Omega_q\, \frac{\sqrt{2mE_q}}{\hbar}\, \delta(E_q - E_k)\, |f(\vec{k},\vec{q})|^2$$

No $|\vec{q}|$ or $|\vec{k}|$ dependence in $|f(\vec{k},\vec{q})|^2$?

$$= \frac{1}{4\pi\hbar^2} \int d\Omega_q \cdot k\, |f(\vec{k},\vec{q})|^2$$

$$= \frac{k}{4\pi\hbar^2} \int d\Omega_q\, |f(\vec{k},\vec{q})|^2 = \frac{k}{4\pi\hbar^2} \sigma_{tot}(k)$$

No q dependence anymore but k?

Setting $\hbar = 1$ yields the result.

H5) ALTERNATIVE TO b)

b) $$f(\vec{k},\vec{k}') = -\frac{m}{2\pi\hbar^2}\langle \vec{k}'|T|\vec{k}\rangle \qquad (1)$$

$$f(\vec{k}',\vec{k})^* = -\frac{m}{2\pi\hbar^2}\langle \vec{k}'|T^+|\vec{k}\rangle \qquad (2)$$

$$\langle \vec{k}'|R|\vec{k}\rangle = -2\pi i\,\delta(E_{k'}-E_k)\langle \vec{k}'|T|\vec{k}\rangle \qquad (3)$$

$$\langle \vec{k}|R^+|\vec{k}'\rangle = 2\pi i\,\delta(E_{k'}-E_k)\langle \vec{k}|T^+|\vec{k}'\rangle \qquad (4)$$

$$\langle \vec{k}'|R+R^+|\vec{k}\rangle = \langle \vec{k}'|-R^+R|\vec{k}\rangle$$

$$= \langle \vec{k}'|R|\vec{k}\rangle + \langle \vec{k}'|R^+|\vec{k}\rangle$$

$$\overset{(3),(4)}{=} -2\pi i\,\delta(E_{k'}-E_k)\langle \vec{k}'|T|\vec{k}\rangle + 2\pi i\,\delta(E_{k'}-E_k)\langle \vec{k}'|T^+|\vec{k}\rangle$$

$$\Leftrightarrow \quad \langle \vec{k}'|T-T^+|\vec{k}\rangle = \left(2\pi i\,\delta(E_{k'}-E_k)\right)^{-1}\langle \vec{k}'|R^+R|\vec{k}\rangle \qquad (5)$$

$$f(\vec{k},\vec{k}') - f(\vec{k}',\vec{k})^* \overset{(1)(2)}{=} -\frac{m}{2\pi\hbar^2}\left\{\langle \vec{k}'|T|\vec{k}\rangle - \langle \vec{k}'|T^+|\vec{k}\rangle\right\}$$

$$\overset{(5)}{=} -\frac{m}{2\pi\hbar^2}\left\{\left(2\pi i\,\delta(E_{k'}-E_k)\right)^{-1}\langle \vec{k}'|R^+R|\vec{k}\rangle\right\}$$

$$= -\frac{m}{2\pi\hbar^2}\left(\frac{1}{2\pi i}\right)\left\{\delta^{-1}(E_{k'}-E_k)\int\frac{d^3q}{(2\pi)^3}\langle \vec{k}'|R^+|\vec{q}\rangle\langle \vec{q}|R|\vec{k}\rangle\right\}$$

$$= -\frac{m}{2\pi\hbar^2}\left(\frac{1}{2\pi i}\right)\left\{\int\frac{d^3q}{(2\pi)^3}\delta^{-1}(E_{k'}-E_k)\left\{2\pi i\,\delta(E_{k'}-E_q)\langle \vec{k}|T^+|\vec{q}\rangle\right\} \times \left\{-2\pi i\,\delta(E_q-E_k)\langle \vec{q}|T|\vec{k}\rangle\right\}\right\}$$

$\delta(E_{k'}-E_q)\cdot\delta(E_q-E_k)$

$= \delta(E_k-E_q)\,\delta(E_{k'}-E_k)$

$$= -\frac{m}{2\pi\hbar^2}\,\frac{2\pi}{i}\left\{\frac{d^3q}{(2\pi)^3}\delta(E_k-E_q)\langle \vec{k}|T^+|\vec{q}\rangle\langle \vec{q}|T|\vec{k}\rangle\right\}$$

$$= \frac{im}{8\pi^3\hbar^2}\left\{\int d^3q\,\delta(E_k-E_q)\left(\frac{2\pi\hbar^2}{m}\right)^2 f(\vec{k}',\vec{q})^* f(\vec{k},\vec{q})\right\}$$

$$= \frac{i\hbar^2}{2\pi m}\int d^3q\,\delta(E_k-E_q)\,f(\vec{k}',\vec{q})^* f(\vec{k},\vec{q})$$

So I got an $\hbar^2$ and used δ^{-1} in (5) ... ? How to avoid