Disclaimer

The solution at hand was written in the course of the respective class at the University of Bonn. If not stated differently on top of the first page or the following website, the solution was prepared and handed in solely by me, Marvin Zanke. Anything in a different color than the ball pen blue is usually a correction that I or a tutor made. For more information and all my material, check:
https://www.physics-and-stuff.com/

I raise no claim to correctness and completeness of the given solutions! This equally applies to the corrections mentioned above.

25.11.2016 Advanced Quantum Theory 5. Exercise Marvin Zanke

H6) Given $F_g[h] := \int d^dx \int d^dx' \, h(\vec{x}) g(\vec{x},\vec{x}') h(\vec{x}')$

using def. $(D_f F)[h] := \lim_{\epsilon\to 0} \frac{1}{\epsilon}\{F[h+\epsilon f] - F[h]\}$

and $(\delta_y F)[h] := \lim_{\epsilon\to 0}\{(D_{\delta_y^\epsilon} F)[h]\}$

we calculate

$$\delta_y F_g[h] = \lim_{\epsilon\to 0}\{(D_{\delta_y^\epsilon} F_g)[h]\} = \lim_{\epsilon\to 0}\{\lim_{\epsilon'\to 0}\frac{1}{\epsilon'}\{F_g[h+\epsilon'\delta_y^\epsilon] - F_g[h]\}\}$$

$$= \lim_{\epsilon\to 0}\lim_{\epsilon'\to 0}\frac{1}{\epsilon'}\{\int d^dx\int d^dx' (h+\epsilon'\delta_y^\epsilon)(\vec{x}) g(\vec{x},\vec{x}')(h+\epsilon'\delta_y^\epsilon)(\vec{x}') - \int d^dx \int d^dx' \, h(\vec{x}) g(\vec{x},\vec{x}') h(\vec{x}')\}$$

Do those things $\epsilon'^2 \delta_y^\epsilon(\vec{x}) g(\vec{x},\vec{x}')\delta_y^\epsilon(\vec{x}')$ etc. commute? And (*)

$$= \lim_{\epsilon\to 0}\lim_{\epsilon'\to 0}\frac{1}{\epsilon'}\{\int d^dx\int d^dx' [h(\vec{x}) g(\vec{x},\vec{x}')\epsilon'\delta_y^\epsilon(\vec{x}') + \epsilon'\delta_y^\epsilon(\vec{x}) g(\vec{x},\vec{x}') h(\vec{x}') + \epsilon'^2 \delta_y^\epsilon(\vec{x}) g(\vec{x},\vec{x}')\delta_y^\epsilon(\vec{x}')]\}$$

Could you also first do the limit in ϵ'?

$$= \lim_{\epsilon'\to 0}\frac{1}{\epsilon'}\{\int d^dx \, h(\vec{x}) g(\vec{x},\vec{y})\epsilon' + \int d^dx' \epsilon' g(\vec{y},\vec{x}') h(\vec{x}') + \epsilon'^2 g(\vec{y},\vec{y})\}$$

Valid to use the limit ϵ twice: for $g(\vec{x},\vec{y})$ and $g(\vec{y},\vec{y})$?

$$\overset{(*)}{=} \int d^dx \{h(\vec{x}) g(\vec{x},\vec{y}) + g(\vec{y},\vec{x}) h(\vec{x})\}$$

Note: I think you should have had taken the $\epsilon' \to 0$ limit first and only ~~then~~ after the $\epsilon\to 0$ (the order of limits is important!)

H7) $S[x] = \int_{t_0}^{t_1} dt' \, L(x(t'),\dot{x}(t'))$

Where from Lagrangian?

a) $(\delta_t S)[x] = \lim_{\epsilon\to 0}\{(D_{\delta_t^\epsilon} S)[x]\} = \lim_{\epsilon\to 0}\lim_{\epsilon'\to 0}\frac{1}{\epsilon'}\{S[x+\epsilon'\delta_t^\epsilon] - S[x]\}$

$$= \lim_{\epsilon\to 0}\lim_{\epsilon'\to 0}\frac{1}{\epsilon'}\int_{t_0}^{t_1} dt' \{L[(x+\epsilon'\delta_t^\epsilon)(t'),(\dot{x}+\epsilon'\dot{\delta}_t^\epsilon)(t')] - L[x(t'),\dot{x}(t')]\}$$

Why limits $\epsilon\to 0^+$, $\epsilon'\to 0$? No "upper" limit needed?

$$= \lim_{\epsilon\to 0}\lim_{\epsilon'\to 0}\frac{1}{\epsilon'}\int_{t_0}^{t_1} dt' \{L(x(t')+\epsilon'\delta_t^\epsilon(t'), \dot{x}(t')+\epsilon'\dot{\delta}_t^\epsilon(t')) - L(x(t'),\dot{x}(t'))\}$$

b) $Tf(x,y,x_0,y_0) \overset{\text{general form up to 2nd order}}{=} f(x_0,y_0) + (x-x_0)\partial_x f(x_0,y_0) + (y-y_0)\partial_y f(x_0,y_0) + \ldots$

This ∂_x is derivative with respect to 1st variable, not x?

Taylor result from a) around $(x(t'),\dot{x}(t'))$ $\Rightarrow$ $(\delta_t S)[x] = \lim_{\epsilon\to 0}\lim_{\epsilon'\to 0}\frac{1}{\epsilon'}\int dt' \{\partial_x L(x(t'),\dot{x}(t'))\epsilon'\delta_t^\epsilon(t') + \partial_{\dot{x}} L(x(t'),\dot{x}(t'))\epsilon'\dot{\delta}_t^\epsilon(t') + \mathcal{O}(\epsilon'^2)\}$

$$= \lim_{\epsilon\to 0}\int dt' \{\delta_t^\epsilon(t')(\partial_x L)(x(t'),\dot{x}(t')) + \dot{\delta}_t^\epsilon(t')(\partial_{\dot{x}} L)(x(t'),\dot{x}(t'))\}$$

lim inside integral and $\lim_{\epsilon\to 0}\mathcal{O}(\epsilon)=0$?

c) $(\delta_t S)[x] = \lim_{\epsilon\to 0}\int_{t_0}^{t_1} dt' (\partial_x h)(x(t'),\dot{x}(t'))\,\delta_t^\epsilon(t') + \lim_{\epsilon\to 0}\int_{t_0}^{t_1} dt'\,\partial_{\dot{x}} h(x(t'),\dot{x}(t'))\,\dot{\delta}_t^\epsilon(t')$

$$\overset{(5)\text{ and I.b.p.}}{=} \partial_x h(x(t),\dot{x}(t)) + \lim_{\epsilon\to 0}\Big\{\Big[\delta_t^\epsilon(t')\,\partial_{\dot{x}}(h(x(t'),\dot{x}(t')))\Big]\Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} dt'\,\partial_{t'}\partial_{\dot{x}} h(x(t'),\dot{x}(t'))\,\delta_t^\epsilon(t')\Big\}$$

why $\hat{=}0$?

Is it ∂_t or $\partial_{t'}$ here?

$$= \partial_x h(x(t),\dot{x}(t)) - \partial_t\partial_{\dot{x}} h(x(t),\dot{x}(t))$$

what happened to the boundary term? why does it vanish?

$$= (\partial_x h - \partial_t\partial_{\dot{x}} h)(x(t),\dot{x}(t))$$

d) $(\partial_x L - \partial_t\partial_{\dot{x}} h)(x_{cl}(t),\dot{x}_{cl}(t)) = \delta_t S[x_{cl}] = \lim_{\epsilon\to 0}\{(D_{\delta_t^\epsilon} S)[x_{cl}]\}$

NICE!!

$$= \lim_{\epsilon\to 0} 0 = 0 \quad \text{as required,}$$

(10) holds $\forall x \in \mathbb{R}$

Generalized to higher dim. and derivatives!

What is h integrated over time?