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01.12.2016 Advanced Quantum theory Exercise 6 Marvin Zanke

H8) $S[x] := \int_{t_a}^{t_b} dt\, h(x(t), \dot{x}(t))$, $h(x, \dot{x}) := \frac{m}{2}\dot{x}^2 - \frac{m\omega^2}{2}x^2$

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a) H7c): $(\delta_t S)[x] = (\partial_x h - \partial_t \partial_{\dot{x}} h)(x(t), \dot{x}(t))$

$$\Rightarrow (\delta_{t_1} S)[x] = (\partial_x h - \partial_{t_1}\partial_{\dot{x}} h)(x(t_1), \dot{x}(t_1))$$
$$= -m\omega^2 x(t_1) - \partial_{t_1} m\dot{x}(t_1) = -m\omega^2 x(t_1) - m\ddot{x}(t_1)$$

Show for a special x?

P.6 (b)(i): $\bar{\delta}_{t_1}[x] := x(t_1)$

$\ddot{x} \hat{=} \partial_{t_1}^2 x$ not ∂_t^2?

$$\rightsquigarrow (\delta_{t_1} S)[x] = -m\omega^2 \bar{\delta}_{t_1}[x] - m\,\bar{\delta}_{t_1}[\ddot{x}]$$
$$= -m\omega^2 \bar{\delta}_{t_1}[x] - m\,\bar{\delta}_{t_1} \circ \partial_{t_1}[x]$$
$$\Rightarrow (\delta_{t_1} S) = -m\omega^2 \bar{\delta}_{t_1} - m\bar{\delta}_{t_1} \circ \partial_{t_1}$$

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b) $\mathcal{D} := \bar{\delta}_{t_1} \circ \partial_t^n$ is a linear functional, because:

where square brackets/parantheses?

$$\mathcal{D}[h + \alpha f] = \bar{\delta}_{t_1}(\partial_t^n[h + \alpha f]) = \bar{\delta}_{t_1}(\partial_t^n h + \alpha \partial_t^n f)$$
$$= \bar{\delta}_{t_1}(\partial_t^n h) + \alpha \bar{\delta}_{t_1}(\partial_t^n f) = \mathcal{D}[h] + \alpha \mathcal{D}[f]$$

where we used that $\bar{\delta}_{t_1}$ is linear: $\bar{\delta}_{t_1}[h + \alpha f] = (h + \alpha f)(t_1)$

Assume ∂_t^n linear?

and ∂_t^n is linear (w/o proof) $= h(t_1) + \alpha f(t_1)$

$= \bar{\delta}_{t_1}[h] + \alpha \bar{\delta}_{t_1}[f]$

P6(a): F linear $\Rightarrow (D_f F)[h] = F[f]$

$$\Rightarrow \delta_{t_2}(\mathcal{D})[x] = \lim_{\epsilon \to 0}\{(D_{\delta^\epsilon_{t_2}}\mathcal{D})[x]\} = \lim_{\epsilon \to 0}\{\mathcal{D}[\delta^\epsilon_{t_2}]\}$$

lim "inside" derivation?

$$= \lim_{\epsilon \to 0}\{\bar{\delta}_{t_1}(\partial_t^n \delta^\epsilon_{t_2})\} = \lim_{\epsilon \to 0}(\partial_t^n \delta^\epsilon_{t_2}(t_1))$$
$$= \partial_t^n \delta_{t_2}(t_1) = \partial_t^n \delta(t_2 - t_1) = (\partial_t^n \delta)(t_1 - t_2)$$

Constant in x, but why $\partial_t^n \delta(t_1 - t_2) = (\partial_t^n \delta)(t_1 - t_2)$

$\Rightarrow$ Constant in x

P7(a): D_f is linear

$$(\delta_{t_2}\delta_{t_1} S)[x] = \delta_{t_2}(-m\omega^2 \bar{\delta}_{t_1} - m\bar{\delta}_{t_1} \circ \partial_t^2)[x] \lim_{\epsilon \to 0}\{(D_{\delta^\epsilon_{t_2}}(-m\omega^2 \bar{\delta}_{t_1} - m\bar{\delta}_{t_1} \circ \partial_t^2))[x]\}$$
$$= \lim_{\epsilon \to 0}\{(D_{\delta^\epsilon_{t_2}}(-m\omega^2 \bar{\delta}_{t_1}))[x] - (D_{\delta^\epsilon_{t_2}}(m\bar{\delta}_{t_1} \circ \partial_t^2))[x]\}$$
$$= \lim_{\epsilon \to 0}[(-m\omega^2)(D_{\delta^\epsilon_{t_2}} \bar{\delta}_{t_1})[x]] - m \lim_{\epsilon \to 0}\{(D_{\delta^\epsilon_{t_2}}(\bar{\delta}_{t_1} \circ \partial_t^2))[x]\}$$
$$= -m\omega^2 \delta_{t_2}(\bar{\delta}_{t_1})[x] - m\,\delta_{t_2}(\bar{\delta}_{t_1} \circ \partial_t^2)[x]$$

$= -m\omega^2 \delta_{t_1}(\delta_{t_2}) - m\ddot{\delta}(t_1 - t_2)$, where we used what we have

$= -m\omega^2 \delta(t_1 - t_2) - m\ddot{\delta}(t_1 - t_2)$ just proven for $n=0$ and $n=2$ ∇

$\Rightarrow (\delta_{t_2}\delta_t S) = -m\omega^2 \delta(t_1 - t_2) - m\ddot{\delta}(t_1 - t_2)$ and therefore especially constant in $[x]$ ∇

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c) the functional derivative of a constant functional vanishes, because:

$$(\delta_y F)[h] = \lim_{\epsilon \to 0}\{(D_{\delta_y^\epsilon} F)[h]\} = \lim_{\epsilon \to 0}\lim_{\epsilon' \to 0}\frac{1}{\epsilon'}\{F[h + \epsilon'\delta_y^\epsilon] - F[h]\}$$

Use $\bar{y}$ to prove in general?

$$= \lim_{\epsilon \to 0}\lim_{\epsilon' \to 0}\{C - C\} = 0, \text{ where } F[h] = C \text{ for any } h \in V.$$

Therefore it is trivial that

$\delta_{t_3}(\delta_{t_2}\delta_{t_1}S) = 0$, as we showed that $(\delta_{t_2}\delta_{t_1}S)$ is a constant functional

By this, it also follows that

$\delta_{t_n} \dots \delta_{t_2}\delta_{t_1} S = 0$ for $n \geq 3$ and any permutation of these

Do we need to permute all of them?

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d) functional Taylor expansion:

Why for $n=0$ term $F[x]$?

$$F[x+y] = \sum_{n=0}^{\infty} \int_{[t_a,t_b]^n} dt_1 \cdots dt_n \, \frac{(\delta_{t_n} \cdots \delta_{t_1} F)[x]}{n!} \, y(t_1) \cdot \ldots \cdot y(t_n)$$

$$= F[x] + \int_{t_a}^{t_b} dt_1 \, (\delta_{t_1} F)[x] \, y(t_1) + \frac{1}{2} \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \, (\delta_{t_2}\delta_{t_1} F)[x] \, y(t_1) y(t_2) + \ldots$$

what the poll x? around x?

why taylor expansion like this?

$$\leadsto \quad S[x_a + y] = S[x_a] + \int_{t_a}^{t_b} dt_1 \, (\delta_{t_1} S)[x_a] \, y(t_1)$$

$$+ \frac{1}{2} \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \, (\delta_{t_2}\delta_{t_1} S)[x_a] \, y(t_1) y(t_2)$$

$$+ \sum_{n=3}^{\infty} \int_{[t_a,t_b]^n} dt_1 \cdots dt_n \, \underbrace{\frac{(\delta_{t_n} \cdots \delta_{t_1} S)[x_a]}{n!}}_{=0\text{, see (c)}} \, y(t_1) \cdot \ldots \cdot y(t_n)$$

the 2nd term is equal to:

∂x or ∂x_a?

$$\int_{t_a}^{t_b} dt_1 \, (\delta_{t_1} S)[x_a] \, y(t_1) \overset{}{\underset{\text{H7(c)}}{=}} \int_{t_a}^{t_b} dt_1 \left\{ \partial_x h - \partial_t \partial_{\dot{x}} h \right\} (x_a(t_1), \dot{x}_a(t_1))$$

$$= \int_{t_a}^{t_b} dt_1 \, \{0\} = 0$$

$$\Longrightarrow \quad S[x_a + y] = S[x_a] + \frac{1}{2} \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \, (\delta_{t_2}\delta_{t_1} S)[x_a] \, y(t_1) y(t_2)]$$

$$= S[x_a] + \underbrace{\frac{1}{2} \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \left\{ -m\omega^2 \delta(t_1 - t_2) - m \ddot{\delta}(t_1 - t_2) \right\} y(t_1) y(t_2)}_{(*)}$$

$$(*) = \frac{1}{2} \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \left\{ -m\omega^2 \delta(t_1 - t_2) \right\} y(t_1) y(t_2)$$

$$- \frac{1}{2} \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \, m \left(-\partial_{t_2} \dot{\delta}(t_1 - t_2) \right) y(t_1) y(t_2)$$

$$= -\frac{1}{2} \int_{t_a}^{t_b} dt_1 \, m\omega^2 y^2(t_1) + \frac{m}{2} \int_{t_a}^{t_b} dt_1 \, y(t_1) \left\{ \dot{\delta}(t_1 - t_2) y(t_2) \Big|_{t_a}^{t_b} - \int_{t_a}^{t_b} dt_2 \, \dot{\delta}(t_1 - t_2) \dot{y}(t_2) \right\}$$

$y(t_a) = y(t_b) = 0$

still for $\delta(x) \cdot y(x)$?

$$= -\frac{m\omega^2}{2} \int_{t_a}^{t_b} dt_1 \, y^2(t_1) - \frac{m}{2} \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \, \partial_{t_1} \delta(t_1 - t_2) \, y(t_1) \dot{y}(t_2)$$

Vanishing boundary terms

$$= -\frac{m\omega^2}{2} \int_{t_a}^{t_b} dt_1 \, y^2(t_1) - \frac{m}{2} \int_{t_a}^{t_b} dt_2 \, \dot{y}(t_2) \left\{ \delta(t_1 - t_2) y(t_1) - \int_{t_a}^{t_b} dt_1 \, \delta(t_1 - t_2) \dot{y}(t_1) \right\}$$

What if again $-\partial_{t_2} \delta(t_1 - t_2)$

$\longrightarrow -\int dt \, \frac{m\omega^2}{2} y^2(t)$

$- \frac{m}{2} \int dt \, y(t) \ddot{y}(t)$

$$= -\frac{m\omega^2}{2}\int_{t_a}^{t_b} dt\, y^2(t) + \frac{m}{2}\int_{t_a}^{t_b} dt\, \dot{y}^2(t) = \int_{t_a}^{t_b} dt \left\{ -\frac{m\omega^2}{2} y^2(t) + \frac{m}{2}\dot{y}^2(t) \right\}$$

boundary term

$$= \int_{t_a}^{t_b} dt\, L(y(t), \dot{y}(t)) = S[y]$$

$$\Rightarrow S[x_a + y] = S[x_a] + S[y]$$ ☒ 7/7

Why no QM $\hbar$?

Energy operator hermitian?