Disclaimer

Advanced Quantum Theory   Exercise 7   Marvin Zanke

H9) By $\vec{x}$, we always denote the last 3 components of $\underbrace{x}_{4\text{-vector}} =: X$

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a)
$$\phi(x) = \int d^4k\, \delta(k^2 - m^2)\, g(k)\, e^{-i(k\cdot x)}$$
$$= \int d^3k \int dk^0\, \delta(k_\mu k^\mu - m^2)\, g\begin{pmatrix} k^0 \\ \vec{k} \end{pmatrix} e^{-i(k_\mu x^\mu)}$$
$$= \int d^3k \int dk^0\, \delta(k_0 k^0 + k_i k^i - m^2)\, g\begin{pmatrix} k^0 \\ \vec{k} \end{pmatrix} e^{-i(k_0 x^0 + k_i x^i)}$$
$$= \int d^3k \int dk^0\, \delta\big((k^0)^2 - (k^i k^i + m^2)\big)\, g\begin{pmatrix} k^0 \\ \vec{k} \end{pmatrix} e^{-i(k^0 x^0 - k^i x^i)}$$

$E_k = \sqrt{|\vec{k}|^2 + m^2} = \sqrt{k^i k^i + m^2}$

$$= \int d^3k \int dk^0\, \delta((k^0)^2 - E_k^2)\, g\begin{pmatrix} k^0 \\ \vec{k} \end{pmatrix} e^{-i(k^0 x^0 - k^i x^i)}$$

δ-Distribution for compositions $\delta(g(x))$

$$= \int d^3k \int dk^0\, \frac{1}{2|E_k|} \left\{ \delta(k^0 - E_k) + \delta(k^0 + E_k) \right\} g\begin{pmatrix} k^0 \\ \vec{k} \end{pmatrix} e^{-i(k^0 x^0 - k^i x^i)}$$

$$\overset{E_k > 0}{=} \int d^3k\, \frac{1}{2E_k} \left\{ g\begin{pmatrix} E_k \\ \vec{k} \end{pmatrix} e^{-i(E_k x^0 - \vec{k}\cdot\vec{x})} + g\begin{pmatrix} -E_k \\ \vec{k} \end{pmatrix} e^{-i(-E_k x^0 - \vec{k}\cdot\vec{x})} \right\}$$

$x^0 = ct \overset{c=1}{=} t$

$$= \int d^3k\, \frac{g\begin{pmatrix} E_k \\ \vec{k} \end{pmatrix}}{2E_k}\, e^{i(\vec{k}\cdot\vec{x} - E_k t)} + \int d^3k\, \frac{g\begin{pmatrix} -E_k \\ \vec{k} \end{pmatrix}}{2E_k}\, e^{i(\vec{k}\cdot\vec{x} + E_k t)}$$

$$= \int d^3k\, \frac{g_+(\vec{k})}{2E_k}\, e^{i(\vec{k}\cdot\vec{x} - E_k t)} + \int d^3k\, \frac{g_-(\vec{k})}{2E_k}\, e^{i(\vec{k}\cdot\vec{x} + E_k t)}$$

using $g_\pm(\vec{k}) := g(\pm E_k, \vec{k})$

$$= \phi_+(x) + \phi_-(x)$$

using $\phi_\pm(x) := \int d^3k\, \frac{g_\pm(\vec{k})}{2E_k}\, e^{i(\vec{k}\cdot\vec{x} \mp E_k t)}$

b) $j^\mu := \frac{i}{2m} \left\{ \phi^*(\partial^\mu \phi) - (\partial^\mu \phi^*)\phi \right\}$ with $(\Box + m^2)\phi = 0 \Rightarrow (\Box + m^2)\phi^* = 0$

$\Leftrightarrow \partial_\mu \partial^\mu \phi = -m^2\phi, \quad \partial_\mu\partial^\mu \phi^* = -m^2 \phi^*$

$$\partial_\mu j^\mu = \frac{i}{2m} \left\{ \left[(\partial_\mu \phi^*)(\partial^\mu \phi) + \phi^*(\partial_\mu \partial^\mu \phi)\right] - \left[(\partial_\mu \partial^\mu \phi^*)\phi - (\partial^\mu \phi^*)(\partial_\mu \phi)\right] \right\}$$
$$= \frac{i}{2m} \left\{ (\partial_\mu \phi^*)(\partial^\mu \phi) - (\partial^\mu \phi^*)(\partial_\mu \phi) + \underbrace{\phi^*(-m^2\phi) - (-m^2\phi^*)\phi}_{-m^2\phi^*\phi + m^2\phi^*\phi = 0} \right\}$$
$$= \frac{i}{2m} \left\{ (\partial_\mu \phi^*)(\partial^\mu \phi) - (\partial^\mu \phi^*)(\partial_\mu \phi) \right\}$$
$$= \frac{i}{2m} \left\{ (\partial_\mu \phi^*)(\eta^{\mu\nu} \partial_\nu \phi) - (\partial^\mu \phi^*)(\partial_\mu \phi) \right\}$$
$$= \frac{i}{2m} \left\{ (\partial^\nu \phi^*)(\partial_\nu \phi) - (\partial^\mu \phi^*)(\partial_\mu \phi) \right\} = 0$$

How to denote that d^3k is only the spatial space?

$k_i k_i = k^i k^i$! $k_0 x_0$ or $k^0 x^0$?

$p \hat{=} k$? $(\to k_0 \hat{=} E?)$

prove this by $\delta(x^2 - a^2) = \delta((x-a)(x+a))$?

Important whether $g(\pm E_k, \vec{k})$ or $g\begin{pmatrix} \pm E_k \\ \vec{k} \end{pmatrix}$?

$\partial_\mu \partial^\mu$ no effect with c.c!

$\partial_\mu \partial^\mu \phi$ scalar number?

Formal proof with those $\eta^{\mu\nu}$? Or why can operate to the left side?

c) $\|\phi\|_c^2 := \int d^3x\, j^0(t,\vec{x}) = \frac{i}{2m}\int d^3x \left[\phi^*(t,\vec{x})(\partial^0\phi)(t,\vec{x}) - (\partial^0\phi^*)(t,\vec{x})\phi(t,\vec{x})\right]$

$$\Rightarrow \underset{\partial_t}{\overset{\|}{\partial^0}} \|\phi\|_c^2 = \int d^3x\, \partial_0 j^0(t,\vec{x}) \overset{b)}{=} -\int d^3x\, \partial_i j^i(t,\vec{x}) = -\int d^3x\, \vec{\nabla}\vec{j}(t,\vec{x})$$

$$\overset{\text{Gauss theorem}}{=} -\int_S d\vec{A}\, \vec{j}(t,\vec{x}) = 0$$ as $S := \{x \in \mathbb{R}^3 \mid \|x\|_2 = I,\ I = \lim_{x\to\infty} x\}$ and ϕ needs to be normalizable, so that it has to be rapidly decreasing for large $|\vec{x}|$. Therefore $\vec{j} = \frac{i}{2m}[\phi^*(\vec{\nabla}\phi) - (\vec{\nabla}\phi^*)\phi]$ has to be as well.

$\Rightarrow \partial_t \|\phi\|_c^2 = 0$ and $\|\phi\|_c^2$ time independent

Normalization ϕ? Index "c"?

$\partial^0\phi: \mathbb{R}^4 \to \mathbb{C}$?

Move ∂^0 into integral?

$\partial_i j^i = \pm\vec{\nabla}\vec{j}$?

How to denote that sphere?

$\vec{j}$ rapidly decreasing

For different way, see 2^{nd} sheet

d) $\|\Phi\|_c^2\big|_{t=0} = \frac{i}{2m}\int d^3x \left[\phi^*(0,\vec{x})(\partial^0\phi)(0,\vec{x}) - (\partial^0\phi^*)(0,\vec{x})\phi(0,\vec{x})\right]$

From a) we got: $\phi^*(0,\vec{x}) = \phi_+^*(0,\vec{x}) + \phi_-^*(0,\vec{x})$

and $(\partial^0\phi)(0,\vec{x}) = \partial_t\phi_+(0,\vec{x}) + \partial_t\phi_-(0,\vec{x})$

$= -iE_k(\phi_+ - \phi_-)$

because $\partial_t\phi_\pm(t,\vec{x})\big|_{t=0} = \int d^3k \frac{g_\pm(\vec{k})}{2E_k}\partial_t e^{i(\vec{k}\cdot\vec{x} \mp E_k t)}\big|_{t=0}$

$= \mp iE_k\,\phi_\pm(0,\vec{x})$ (analogue for c.c.)

New calculation for c.c. or just c.c. equation?

$$\rightsquigarrow \|\Phi\|_c^2\big|_{t=0} = \frac{i}{2m}\int d^3x \Big\{(\phi_+^*(0,\vec{x}) + \phi_-^*(0,\vec{x}))\cdot(-iE_k)(\phi_+(0,\vec{x}) - \phi_-(0,\vec{x}))$$
$$- (iE_k)(\phi_+^*(0,\vec{x}) - \phi_-^*(0,\vec{x}))(\phi_+(0,\vec{x}) + \phi_-(0,\vec{x}))\Big\}$$

$$= \frac{E_k}{2m}\int d^3x \Big\{|\phi_+(0,\vec{x})|^2 - |\phi_-(0,\vec{x})|^2 - \phi_+^*(0,\vec{x})\phi_-(0,\vec{x})$$
$$+ \phi_-^*(0,\vec{x})\phi_+(0,\vec{x}) + |\phi_+(0,\vec{x})|^2 - |\phi_-(0,\vec{x})|^2$$
$$+ \phi_+^*(0,\vec{x})\phi_-(0,\vec{x}) - \phi_-^*(0,\vec{x})\phi_+(0,\vec{x})\Big\}$$

$$= \frac{E_k}{2m}\int d^3x \left\{2|\phi_+(0,\vec{x})|^2 - 2|\phi_-(0,\vec{x})|^2\right\}$$

$$= \frac{E_k}{m}\int d^3x \left\{\phi_+(0,\vec{x})\phi_+^*(0,\vec{x}) - \phi_-(0,\vec{x})\phi_-^*(0,\vec{x})\right\} \quad (*)$$

Now we calculate:

Pull * into the integral?

$$\int d^3x\, \phi_\pm(0,\vec{x})\phi_\pm^*(0,\vec{x}) = \int d^3x \left\{\int d^3k \frac{g_\pm(\vec{k})}{2E_k} e^{i(\vec{k}\cdot\vec{x})} \cdot \int d^3k' \frac{g_\pm^*(\vec{k}')}{2E_{k'}} e^{-i(\vec{k}'\cdot\vec{x})}\right.$$

$$= \int d^3x \iint d^3k\, d^3k' \frac{g_\pm(\vec{k})g_\pm^*(\vec{k}')}{4E_kE_{k'}} e^{i\vec{x}(\vec{k}-\vec{k}')}$$

$$= \iint d^3k\, d^3k' \frac{g_\pm(\vec{k})g_\pm^*(\vec{k}')}{4E_kE_{k'}} \underbrace{\int d^3x\, e^{i\vec{x}(\vec{k}-\vec{k}')}}_{\text{representation of the } \delta\text{-distribution in sense of Fourier-trafo's}} \Rightarrow \overset{d^3x}{=} (2\pi)^3\delta(\vec{k}-\vec{k}')$$

Could there also be an $f(\vec{x})$ in integral? $\int d^3x\, f(\vec{x})e^{i\vec{x}(\vec{k}-\vec{k}')}$ and still work?

$$= \iint d^3k\, d^3k'\, \delta(\vec{k}-\vec{k}')(2\pi)^3 \frac{g_\pm(\vec{k})g_\pm^*(\vec{k}')}{4E_kE_{k'}}$$

$$= \int d^3k \frac{|g_\pm(\vec{k})|^2}{4E_k^2}(2\pi)^3$$

Now, for (*) we get:

$$\|\phi\|_c^2\big|_{t=0} = \frac{E_k}{m}\int d^3x \left\{ \phi_+(0,\vec{x})\,\phi_+^*(0,\vec{x}) - \phi_-(0,\vec{x})\,\phi_-^*(0,\vec{x}) \right\}$$

$$= \frac{E_k}{m}\left\{ \int d^3k \frac{(2\pi)^3}{4E_k^2}|g_+(\vec{k})|^2 - \int d^3k \frac{(2\pi)^3}{4E_k^2}|g_-(\vec{k})|^2 \right\}$$

$$= \frac{1}{2m}\int d^3k \frac{(2\pi)^3}{2E_k}|g_+(\vec{k})|^2 - \frac{1}{2m}\int d^3k \frac{(2\pi)^3}{2E_k}|g_-(\vec{k})|^2$$

$$= \|\phi_+\|_c^2 + \|\phi_-\|_c^2 \quad \text{with} \quad \|\phi_\pm\|_c^2 := \pm\frac{1}{2m}\int d^3k \frac{(2\pi)^3}{2E_k}|g_\pm(\vec{k})|^2$$

Why is E_k outside of the integral?

e) $$\|\phi_\pm\|_c^2 \overset{!}{=} \pm\frac{(2\pi)^3}{2m}\int d^4k\, \underbrace{\delta(k^2-m^2)}_{\delta(k_0^2-\vec{k}^2-m^2) = \delta(E^2-(k_0^2-m^2))}\,\theta(\pm k^0)\,|g(k)|^2$$

Not starting with the result?

$$= \pm\frac{(2\pi)^3}{2m}\int d^4k\, \underbrace{\delta(k_0^2-E_k^2)}_{\frac{1}{2E_k}\{\delta(k_0-E_k)+\delta(k_0+E_k)\}}\,\underbrace{\theta(\pm k^0)}_{\text{"+" : } k^0>0;\ \text{"−" : } k^0<0}\,|g(k)|^2$$

$$= \pm\frac{(2\pi)^3}{2m}\left\{ \int d^4k \frac{1}{2E_k}\overbrace{\delta(k_0-E_k)}^{k_0 \hat{=} E_k > 0}\,\theta(\pm k^0)\,|g(k)|^2 + \int d^4k \frac{1}{2E_k}\underbrace{\delta(k_0+E_k)}_{k_0 = -E_k < 0}\,\theta(\pm k^0)\,|g(k)|^2 \right\}$$

k_0 or k^0?

first term only contributes for "+" and second only for "−"

$$= \pm\frac{(2\pi)^3}{2m}\int d^3k \frac{1}{2E_k}|g_\pm(\vec{k})|^2 = \pm\frac{1}{2m}\int d^3k \frac{(2\pi)^3}{2E_k}|g_\pm(\vec{k})|^2 = \|\phi_\pm\|_c^2$$

d) Alternative way without $E_k = \sqrt{|\vec{k}|^2 + m^2}$ out of integral:

$$\|\phi\|_c^2 \big|_{t=0} = \frac{i}{2m} \int d^3x \left\{ \phi^*(0,\vec{x}) (\partial^0 \phi)(0,\vec{x}) - (\partial^0 \phi^*)(0,\vec{x})\, \phi(0,\vec{x}) \right\}$$

From a), we get: $\phi(0,\vec{x}) = \phi_+(0,\vec{x}) + \phi_-(0,\vec{x})$ (analogue for c.c.)

and
$$(\partial^0 \phi)(0,\vec{x}) = \partial_t \phi_+(0,\vec{x}) + \partial_t \phi_-(0,\vec{x})$$
$$= \int d^3k\,(-i)\,\frac{g_+(\vec{k})}{2} e^{i\vec{k}\cdot\vec{x}} + \int d^3k\,(i)\,\frac{g_-(\vec{k})}{2} e^{i\vec{k}\cdot\vec{x}}$$
$$=: -i(\tilde{\phi}_+ - \tilde{\phi}_-) \quad \text{(analogue for c.c.)}$$

all arguments $(0,\vec{x})$ are left out

$$\rightsquigarrow \|\phi\|_c^2 \big|_{t=0} = \frac{i}{2m} \int d^3x \left\{ (\phi_+^* + \phi_-^*)(-i)(\tilde{\phi}_+ - \tilde{\phi}_-) - (i)(\tilde{\phi}_+^* - \tilde{\phi}_-^*)(\phi_+ + \phi_-) \right\}$$

$$= \frac{1}{2m} \int d^3x \left\{ \phi_+^* \tilde{\phi}_+ - \phi_+^* \tilde{\phi}_- + \phi_-^* \tilde{\phi}_+ - \phi_-^* \tilde{\phi}_- + \tilde{\phi}_+^* \phi_+ + \tilde{\phi}_+^* \phi_- - \tilde{\phi}_-^* \phi_+ - \tilde{\phi}_-^* \phi_- \right\}$$

$$\begin{aligned}
= \frac{1}{2m} \int d^3x \Big\{ & \iint d^3k\, d^3k' \frac{g_+^*(\vec{k})}{2E_k} e^{-i\vec{k}\cdot\vec{x}} \frac{g_+(\vec{k}')}{2} e^{i\vec{k}'\cdot\vec{x}} \\
& - \iint d^3k\, d^3k' \frac{g_+^*(\vec{k})}{2E_k} e^{-i\vec{k}\cdot\vec{x}} \frac{g_-(\vec{k}')}{2} e^{i\vec{k}'\cdot\vec{x}} \\
& + \iint d^3k\, d^3k' \frac{g_-^*(\vec{k})}{2E_k} e^{-i\vec{k}\cdot\vec{x}} \frac{g_+(\vec{k}')}{2} e^{i\vec{k}'\cdot\vec{x}} \\
& - \iint d^3k\, d^3k' \frac{g_-^*(\vec{k})}{2E_k} e^{-i\vec{k}\cdot\vec{x}} \frac{g_-(\vec{k}')}{2} e^{i\vec{k}'\cdot\vec{x}} \\
& + \iint d^3k\, d^3k' \frac{g_+^*(\vec{k})}{2} e^{-i\vec{k}\cdot\vec{x}} \frac{g_+(\vec{k}')}{2E_{k'}} e^{i\vec{k}'\cdot\vec{x}} \\
& + \iint d^3k\, d^3k' \frac{g_+^*(\vec{k})}{2} e^{-i\vec{k}\cdot\vec{x}} \frac{g_-(\vec{k}')}{2E_{k'}} e^{i\vec{k}'\cdot\vec{x}} \\
& - \iint d^3k\, d^3k' \frac{g_-^*(\vec{k})}{2} e^{-i\vec{k}\cdot\vec{x}} \frac{g_+(\vec{k}')}{2E_{k'}} e^{i\vec{k}'\cdot\vec{x}} \\
& - \iint d^3k\, d^3k' \frac{g_-^*(\vec{k})}{2} e^{-i\vec{k}\cdot\vec{x}} \frac{g_-(\vec{k}')}{2E_{k'}} e^{i\vec{k}'\cdot\vec{x}} \Big\}
\end{aligned}$$

$$\begin{aligned}
= \frac{1}{2m} \iint d^3k\, d^3k' \Big\{ & \frac{g_+^*(\vec{k})\, g_+(\vec{k}')}{4E_k} \int d^3x\, e^{i\vec{x}(\vec{k}'-\vec{k})} - \frac{g_+^*(\vec{k})\, g_-(\vec{k}')}{4E_k} \int d^3x\, e^{i\vec{x}(\vec{k}'-\vec{k})} \\
& + \frac{g_-^*(\vec{k})\, g_+(\vec{k}')}{4E_k} \int d^3x\, e^{i\vec{x}(\vec{k}'-\vec{k})} - \frac{g_-^*(\vec{k})\, g_-(\vec{k}')}{4E_k} \int d^3x\, e^{i\vec{x}(\vec{k}'-\vec{k})} \\
& + \frac{g_+^*(\vec{k})\, g_+(\vec{k}')}{4E_{k'}} \int d^3x\, e^{i\vec{x}(\vec{k}'-\vec{k})} + \frac{g_+^*(\vec{k})\, g_-(\vec{k}')}{4E_{k'}} \int e^{i\vec{x}(\vec{k}'-\vec{k})} d^3x \\
& - \frac{g_-^*(\vec{k})\, g_+(\vec{k}')}{4E_{k'}} \int e^{i\vec{x}(\vec{k}'-\vec{k})} d^3x - \frac{g_-^*(\vec{k})\, g_-(\vec{k}')}{4E_{k'}} \int e^{i\vec{x}(\vec{k}'-\vec{k})} d^3x
\end{aligned}$$

$\frac{1}{2\pi} \int dx\, e^{ix(k-k')} = \delta(k-k')$

$$\begin{aligned}
\downarrow \\
= \frac{(2\pi)^3}{2m} \int d^3k \Big\{ & \frac{|g_+(\vec{k})|^2}{4E_k} - \frac{g_+^*(\vec{k})\, g_-(\vec{k})}{4E_k} + \frac{g_-^*(\vec{k})\, g_+(\vec{k})}{4E_k} - \frac{|g_-(\vec{k})|^2}{4E_k} \\
& + \frac{|g_+(\vec{k})|^2}{4E_k} + \frac{g_+^*(\vec{k})\, g_-(\vec{k})}{4E_k} - \frac{g_-^*(\vec{k})\, g_+(\vec{k})}{4E_k} - \frac{|g_-(\vec{k})|^2}{4E_k} \Big\}
\end{aligned}$$

$$= \frac{1}{2m}(2\pi)^3 \int d^3k \left\{ \frac{|g_+(\vec{k})|^2}{2E_k} - \frac{|g_-(\vec{k})|^2}{2E_k} \right\}$$

$$= \frac{1}{2m} \int d^3k \, \frac{(2\pi)^3}{2E_k} |g_+(\vec{k})|^2 - \frac{1}{2m} \int d^3k \, \frac{(2\pi)^3}{2E_k} |g_-(\vec{k})|^2$$

$$= \|\phi_+\|_c^2 + \|\phi_-\|_c^2 \quad \text{with} \quad \|\phi_\pm\|_c^2 := \pm \frac{1}{2m} \int d^3k \, \frac{(2\pi)^3}{2E_k} |g_\pm(\vec{k})|^2$$