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12.12.2016 Advanced Quantum theory 8. Exercise Marvin Zanke

20/20

H10) $\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\mathbb{1}_4$ (9)

$g_{\mu\nu}g^{\mu\nu} = 4$ or $4\mathbb{1}_4$? $g_{\mu\nu}g^{\mu\nu} = g_{\mu\nu}$ or what?

(i) $\gamma_\mu\gamma^\mu = g_{\mu\nu}\gamma^\nu\gamma^\mu = g_{\mu\nu}\left(\frac{1}{2}\gamma^\nu\gamma^\mu + \frac{1}{2}\gamma^\nu\gamma^\mu\right) = g_{\mu\nu}\left(\frac{1}{2}\gamma^\nu\gamma^\mu\right) + g_{\nu\mu}\left(\frac{1}{2}\gamma^\mu\gamma^\nu\right)$

$\overset{g_{\mu\nu}=g_{\nu\mu}}{=} g_{\mu\nu}\left(\frac{1}{2}\gamma^\nu\gamma^\mu\right) + g_{\mu\nu}\left(\frac{1}{2}\gamma^\mu\gamma^\nu\right) = \frac{1}{2}g_{\mu\nu}\left(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\right)$

$= \frac{1}{2}g_{\mu\nu}\cdot 2g^{\mu\nu}\mathbb{1}_4 = g_{\mu\nu}g^{\mu\nu}\mathbb{1}_4 = 4\mathbb{1}_4$

$\mathbb{1}_4$ and γ^μ commute?

(ii) $\gamma_\mu\gamma^\nu\gamma^\mu \overset{(9)}{=} \gamma_\mu\left(2g^{\mu\nu}\mathbb{1}_4 - \gamma^\mu\gamma^\nu\right) \overset{(i)}{=} 2\gamma^\nu\mathbb{1}_4 - 4\mathbb{1}_4\gamma^\nu$

$= -2\gamma^\nu\mathbb{1}_4 = -2\gamma^\nu$

What is γ_μ if γ^μ are the matrices from lecture?

(iii) Remark: $\{\gamma^\nu, \gamma^\lambda\} = \gamma^\nu\gamma^\lambda + \gamma^\lambda\gamma^\nu = 2g^{\nu\lambda}\mathbb{1}_4$

$\{\gamma^\lambda, \gamma^\nu\} = \gamma^\lambda\gamma^\nu + \gamma^\nu\gamma^\lambda = 2g^{\lambda\nu}\mathbb{1}_4$

What for all those properties (i) - (iv)?

$\rightsquigarrow \gamma_\mu\gamma^\nu\gamma^\lambda\gamma^\mu \overset{(9)}{=} \gamma_\mu\gamma^\nu\left(2g^{\lambda\mu}\mathbb{1}_4 - \gamma^\mu\gamma^\lambda\right) \overset{(ii)}{=} 2g^{\lambda\mu}\gamma_\mu\gamma^\nu\mathbb{1}_4 - \gamma_\mu\gamma^\nu\gamma^\mu\gamma^\lambda$

$\overset{\mathbb{1}_4 \text{ no effect on } \gamma^\mu}{=} 2\gamma^\lambda\gamma^\nu - \left(-2\gamma^\nu\gamma^\lambda\right) = 2\left(\gamma^\lambda\gamma^\nu + \gamma^\nu\gamma^\lambda\right) = 2\{\gamma^\lambda, \gamma^\nu\}$

$= 4g^{\lambda\nu}\mathbb{1}_4$

$g^{\rho\mu}\gamma_\mu = \gamma^\rho$?

(iv) $\gamma_\mu\gamma^\nu\gamma^\lambda\gamma^\rho\gamma^\mu \overset{(9)}{=} \gamma_\mu\gamma^\nu\gamma^\lambda\left(2g^{\rho\mu}\mathbb{1}_4 - \gamma^\mu\gamma^\rho\right) = 2g^{\rho\mu}\gamma_\mu\gamma^\nu\gamma^\lambda\mathbb{1}_4 - \gamma_\mu\gamma^\nu\gamma^\lambda\gamma^\mu\gamma^\rho$

$= 2\gamma^\rho\gamma^\nu\gamma^\lambda - \left(4g^{\lambda\nu}\mathbb{1}_4\right)\gamma^\rho = 2\gamma^\rho\left(2g^{\nu\lambda}\mathbb{1}_4 - \gamma^\lambda\gamma^\nu\right) - 4g^{\lambda\nu}\gamma^\rho$

$= 4\gamma^\rho g^{\nu\lambda} - 2\gamma^\rho\gamma^\lambda\gamma^\nu - 4g^{\lambda\nu}\gamma^\rho = -2\gamma^\rho\gamma^\lambda\gamma^\nu$

Why not keep in this form?

(v) $\mathrm{Tr}(\gamma^\mu\gamma^\nu) \overset{(9)}{=} \mathrm{Tr}\left(2g^{\mu\nu}\mathbb{1}_4 - \gamma^\nu\gamma^\mu\right) \overset{P1a)}{=} 2g^{\mu\nu}\,\mathrm{Tr}(\mathbb{1}_4) - \mathrm{Tr}(\gamma^\nu\gamma^\mu)$

$\overset{P1b)}{=} 2g^{\mu\nu}\cdot 4 - \mathrm{Tr}(\gamma^\mu\gamma^\nu) = 8g^{\mu\nu} - \mathrm{Tr}(\gamma^\mu\gamma^\nu)$

$\Leftrightarrow 2\,\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 8g^{\mu\nu}$

$\Leftrightarrow \mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$

why do $\gamma^\mu p_\mu$ commute? p_μ is scalar, but operator?

(vi) $\mathrm{Tr}(\not{p}\not{k}) = \mathrm{Tr}(\gamma^\mu p_\mu\gamma^\nu k_\nu) = \mathrm{Tr}(p_\mu\gamma^\mu\gamma^\nu k_\nu) \overset{P1a)}{=} p_\mu\,\mathrm{Tr}(\gamma^\mu\gamma^\nu)\,k_\nu$

$\overset{(v)}{=} p_\mu\left(4g^{\mu\nu}\right)k_\nu = 4p^\nu k_\nu = 4\,(p\cdot k)$

(vii) Remark: $\{\gamma^\mu,\gamma^\nu\}=0$ for $\mu\neq\nu \Leftrightarrow \gamma^\mu\gamma^\nu=-\gamma^\nu\gamma^\mu$

$\gamma_5 := i\gamma^0\gamma^1\gamma^2\gamma^3$

$$\rightsquigarrow \{\gamma_5,\gamma^\mu\} = i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu + i\gamma^\mu\gamma^0\gamma^1\gamma^2\gamma^3 = i(\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu+\gamma^\mu\gamma^0\gamma^1\gamma^2\gamma^3)$$

$= \Big\{$

- $\underline{\mu=0}$: $i(\gamma^0\gamma^1\gamma^2(-\gamma^0\gamma^3)+\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3)$

  $\overset{(\gamma^0)^2=\mathbb{1}}{=}$ (ok, not needed) $i(\gamma^0\gamma^1\gamma^0\gamma^2\gamma^3+(\gamma^0)^2\gamma^1\gamma^2\gamma^3)$

  $= i(\gamma^0(-\gamma^0\gamma^1)\gamma^2\gamma^3+(\gamma^0)^2\gamma^1\gamma^2\gamma^3) = 0$

- $\underline{\mu=1}$: $i(\gamma^0\gamma^1\gamma^2(-\gamma^1\gamma^3)+(-\gamma^0\gamma^1)\gamma^1\gamma^2\gamma^3)$

  $= i(\gamma^0\gamma^1(\gamma^1\gamma^2)\gamma^3-\gamma^0\gamma^1\gamma^1\gamma^2\gamma^3)=0$

- $\underline{\mu=2}$: $i(\gamma^0\gamma^1\gamma^2(-\gamma^2\gamma^3)+(-\gamma^0\gamma^2)\gamma^1\gamma^2\gamma^3)$

  $= i(-\gamma^0\gamma^1\gamma^2\gamma^2\gamma^3+\gamma^0\gamma^1\gamma^2\gamma^2\gamma^3)=0$

- $\underline{\mu=3}$: $i(\gamma^0\gamma^1\gamma^2\gamma^3\gamma^3+(-\gamma^0\gamma^3)\gamma^1\gamma^2\gamma^3)$

  $= i(\gamma^0\gamma^1\gamma^2\gamma^3\gamma^3+\gamma^0\gamma^1\gamma^3\gamma^2\gamma^3)$

  $= i(\gamma^0\gamma^1\gamma^2\gamma^3\gamma^3+\gamma^0\gamma^1(\gamma^2\gamma^3)\gamma^3)=0$

Nicer way w/o different cases?

(viii) $Tr(\gamma_5) = Tr(\gamma_5\gamma^\mu(\gamma^\mu)^{-1}) \overset{P1b)}{=} Tr((\gamma^\mu)^{-1}\gamma_5\gamma^\mu) \overset{(vii)}{=} Tr((\gamma^\mu)^{-1}(\gamma^\mu\gamma_5))$

$= -Tr(\gamma_5) \longrightarrow Tr(\gamma_5)=0$ Nice!

How do we know that γ^μ is invertible?

Could also instantly see this with P1c) and (vii) as (vii) proves that γ^μ and γ_5 anticommute. γ^μ is invertible and thereby we can use P1 c).

$(\gamma^0)^2=\mathbb{1},\ \gamma^i(\gamma^i)=\mathbb{1}$

(ix) $(\gamma_5)^2 = \gamma_5(i\gamma^0\gamma^1\gamma^2\gamma^3) = -i\gamma^0\gamma_5\gamma^1\gamma^2\gamma^3 \overset{(\gamma^0)^2=\mathbb{1}_4}{=} (\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3)$

$= \gamma^1\gamma^2(-\gamma^1\gamma^3)\gamma^2\gamma^3 \overset{(\gamma^i)^2=-\mathbb{1}_4}{=} \gamma^1\gamma^1\gamma^2\gamma^3\gamma^2\gamma^3 = -(\gamma^2\gamma^3(-\gamma^3\gamma^2)) = \mathbb{1}_4$

What for (vii)?

(x) $Tr(\gamma^\mu\gamma^\nu\cdots\gamma^\lambda) \overset{(ix)}{=} Tr(\gamma_5\gamma_5\gamma^\mu\gamma^\nu\cdots\gamma^\lambda) \overset{(vii)}{=} Tr((-\gamma_5\gamma^\mu\gamma_5)\gamma^\nu\cdots\gamma^\lambda)$

repeat for all γ's $= Tr((-\gamma_5\gamma^\mu\gamma_5)(-\gamma_5\gamma^\nu\gamma_5)\cdots(-\gamma_5\gamma^\lambda\gamma_5))$

$\overset{(ix)}{=} (-1)^n\, Tr(\gamma_5\gamma^\mu\gamma^\nu\cdots\gamma^\lambda\gamma_5)$, $n := \#\gamma$'s

$\overset{P1b), (ix)}{=} (-1)^n\, Tr(\gamma^\mu\gamma^\nu\cdots\gamma^\lambda)$

For n odd $\Rightarrow Tr(\gamma^\mu\gamma^\nu\cdots\gamma^\lambda) = -Tr(\gamma^\mu\gamma^\nu\cdots\gamma^\lambda) \Leftrightarrow Tr(\gamma^\mu\gamma^\nu\cdots\gamma^\lambda)=0$

b) $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ with $\gamma^0 = \begin{pmatrix} \mathbb{1}_{2\times2} & 0 \\ 0 & -\mathbb{1}_{2\times2} \end{pmatrix} = \beta$

$\gamma^i = \beta\alpha^i \quad , \quad \alpha^i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$

$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

What is meant by standard representation?

$$\rightsquigarrow \gamma^1 = \begin{pmatrix} \mathbb{1}_4 & 0 \\ 0 & -\mathbb{1}_4 \end{pmatrix}\begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} \mathbb{1}_4 & 0 \\ 0 & -\mathbb{1}_4 \end{pmatrix}\begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}$$

$$\gamma^3 = \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}$$

$$\rightsquigarrow \gamma_5 = i\begin{pmatrix} \mathbb{1}_4 & 0 \\ 0 & -\mathbb{1}_4 \end{pmatrix}\begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}$$

Correct order of multiplication important?

$$= i\begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}\begin{pmatrix} -\sigma_2\sigma_3 & 0 \\ 0 & -\sigma_2\sigma_3 \end{pmatrix} = i\begin{pmatrix} 0 & -\sigma_1\sigma_2\sigma_3 \\ -\sigma_1\sigma_2\sigma_3 & 0 \end{pmatrix}$$

$$\sigma_1\sigma_2\sigma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}$$