Disclaimer

The solution at hand was written in the course of the respective class at the University of Bonn. If not stated differently on top of the first page or the following website, the solution was prepared and handed in solely by me, Marvin Zanke. Anything in a different color than the ball pen blue is usually a correction that I or a tutor made. For more information and all my material, check:
https://www.physics-and-stuff.com/

I raise no claim to correctness and completeness of the given solutions! This equally applies to the corrections mentioned above.

23.12.2016 Advanced Quantum Theory Exercise 9 Marvin Zanke

H11) Defining property for gamma matrices: $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}$

a) We want to show that the Weyl representation 20/20

Need to find α^i and β s.t. $\gamma^i = \beta\alpha^i$, $\gamma^0 = \beta$ so dirac eq fulfilled?

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_{2\times2} \\ \mathbb{1}_{2\times2} & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

fulfills this relation.

- $2g^{00}\mathbb{1} \overset{!}{=} \gamma^0\gamma^0 + \gamma^0\gamma^0 = 2\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} = 2\begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} = 2\mathbb{1}$
  $= 2g^{00}\mathbb{1}$

- $2g^{0i}\mathbb{1} \overset{!}{=} \gamma^0\gamma^i + \gamma^i\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$
  $= \begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} + \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
  $= 2g^{0i}\mathbb{1}$

Can you just multiply those blocks of matrices with each other?

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} E & F \\ G & H \end{pmatrix} \overset{?}{=} \begin{pmatrix} AE+BG & AF+BH \\ CE+DG & CF+DH \end{pmatrix}$$

- $2g^{ij}\mathbb{1} \overset{!}{=} \gamma^i\gamma^j + \gamma^j\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$
  $= \begin{pmatrix} -\sigma^i\sigma^j & 0 \\ 0 & -\sigma^i\sigma^j \end{pmatrix} + \begin{pmatrix} -\sigma^j\sigma^i & 0 \\ 0 & -\sigma^j\sigma^i \end{pmatrix}$
  $= -\begin{pmatrix} \{\sigma^i, \sigma^j\} & 0 \\ 0 & \{\sigma^i, \sigma^j\} \end{pmatrix} \underset{\uparrow}{=} -\begin{pmatrix} 2\delta_{ij}\mathbb{1} & 0 \\ 0 & 2\delta_{ij}\mathbb{1} \end{pmatrix}$
  property of Pauli matrices
  $= -2\delta_{ij}\mathbb{1}_{4\times4}$

  $= 2g^{ij}\mathbb{1}$

As $\{\gamma^\mu, \gamma^\nu\} = \{\gamma^\nu, \gamma^\mu\}$ and $g^{\mu\nu} = g^{\nu\mu}$, those matrices fulfill $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}$, $\forall \mu,\nu = 0,\dots,3$

b) $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = i\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix}$

$= i\begin{pmatrix} -\sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix}\begin{pmatrix} -\sigma^2\sigma^3 & 0 \\ 0 & -\sigma^2\sigma^3 \end{pmatrix}$

$= i\begin{pmatrix} \sigma^1\sigma^2\sigma^3 & 0 \\ 0 & -\sigma^1\sigma^2\sigma^3 \end{pmatrix} = i\begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}$

$= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$\gamma^1, \gamma^2, \gamma^3$ exactly the same as in standard representation

as $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\rightsquigarrow \sigma^1\sigma^2\sigma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$

$= \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$

Why $\mathbb{R}^4 \to \mathbb{C}^4$?

H12) For this exercise, we will be using the standard representation of the γ-matrices and $\mathcal{H} := \{ \psi : \mathbb{R}^4 \to \mathbb{C}^4 \mid \psi \text{ spinor} \}$

What does "ψ spinor" mean?

$\hat{C} : \mathcal{H} \to \mathcal{H}$, $\hat{C}\psi := i\gamma^2 \psi^*$

Why $\hat{C}$ defined like this?

a)

$$\gamma^2 \gamma^\mu \gamma^2 \overset{!}{=} (\gamma^\mu)^*$$

$$\gamma^2\gamma^\mu\gamma^2 = \{ 2g^{\mu\nu}\mathbb{1} - \gamma^\nu\gamma^\mu \} \gamma^2 = -2\delta_{\mu 2}\gamma^2 - \gamma^\mu\gamma^2\gamma^2$$

$$\overset{(\gamma^2)^2 = -\mathbb{1}}{=} \gamma^\mu - 2\delta_{\mu 2}\gamma^2 = \begin{cases} \gamma^\mu & \text{for } \mu \neq 2 \\ \gamma^2 - 2\gamma^2 = -\gamma^2 & \text{for } \mu = 2 \end{cases} \qquad (1)$$

<u>AND</u> using $\alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} \rightsquigarrow (\alpha^i)^* = \begin{cases} \alpha^i, & i = 1,3 \\ -\alpha^i, & i = 2 \end{cases}$

with $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Solving without different cases?

$$\rightsquigarrow (\sigma^1)^* = \sigma^1, \quad (\sigma^2)^* = -\sigma^2, \quad (\sigma^3)^* = \sigma^3$$

we find that

$$\gamma^0 = \beta \Rightarrow (\gamma^0)^* = \gamma^0 \quad \text{as } \beta = \begin{pmatrix} \mathbb{1}_{2\times 2} & 0 \\ 0 & -\mathbb{1}_{2\times 2} \end{pmatrix}$$

Could also show this in Weyl repr. → why standard representation?

$$\gamma^i = \beta\alpha^i \Rightarrow (\gamma^i)^* = \beta^*(\alpha^i)^* = \begin{cases} \beta\alpha^i, & i = 1,3 \\ -\beta\alpha^i, & i = 2 \end{cases}$$

$$= \begin{cases} \gamma^i, & i = 1,3 \\ -\gamma^i, & i = 2 \end{cases}$$

and therefore

$$(\gamma^\mu)^* = \begin{cases} \gamma^\mu & \text{for } \mu \neq 2 \\ -\gamma^\mu & \text{for } \mu = 2 \end{cases} \qquad (2)$$

$\rightsquigarrow$ Comparing (1) and (2) yields $(\gamma^\mu)^* = \gamma^2\gamma^\mu\gamma^2$

only c.c. / no comparing?

Besides: $\hat{C}^2\psi = \hat{C}(\hat{C}\psi) = \hat{C}(i\gamma^2\psi^*) = i\gamma^2(i\gamma^2\psi^*)^*$

$$= i\gamma^2(-i(\gamma^2)^*(\psi^*)^*) = \gamma^2(\gamma^2)^*\psi$$

$$= \gamma^2\gamma^2\gamma^2\gamma^2\psi = (-\mathbb{1})(-\mathbb{1})\psi = \psi$$

b) Let ψ be a solution of the Dirac eq with e.m. field and charge e

$\rightsquigarrow$ $(i\not\partial - e\not A - m)\psi(x) = 0$

$\Leftrightarrow (i\partial_\mu\gamma^\mu - eA_\mu\gamma^\mu - m)\psi(x) = 0$ (1)

We want to show that

$(i\not\partial + e\not A - m)(\hat C\psi(x)) = 0$

$\Leftrightarrow (i\not\partial + e\not A - m)(i\gamma^2\psi^*(x)) = 0$

$(1)^* \Leftrightarrow (-i\partial_\mu^*(\gamma^\mu)^* - eA_\mu^*(\gamma^\mu)^* - m)\psi^*(x) = 0$

$\overset{A_\mu \text{ and } \partial_\mu \text{ real and (a)}}{\Longleftrightarrow} (-i\partial_\mu\gamma^2\gamma^\mu\gamma^2 - eA_\mu\gamma^2\gamma^\mu\gamma^2 - m)\psi^*(x) = 0$

$\overset{\text{multiply by } \gamma^2}{\Longleftrightarrow} \gamma^2(-i\partial_\mu\gamma^2\gamma^\mu\gamma^2 - eA_\mu\gamma^2\gamma^\mu\gamma^2 - m)\psi^*(x) = 0$

$\overset{(\gamma^2)^2 = -\mathbb{1},\ \gamma^2 \text{ commutes with } \partial_\mu, A_\mu}{\Longleftrightarrow} (i\partial_\mu\gamma^\mu\gamma^2 + eA_\mu\gamma^\mu\gamma^2 - m\gamma^2)\psi^*(x) = 0$

$\overset{\text{multiply by } i}{\Longleftrightarrow} (i\partial_\mu\gamma^\mu + eA_\mu\gamma^\mu - m)(i\gamma^2\psi^*(x)) = 0$

$\Leftrightarrow (i\not\partial + e\not A - m)(\hat C\psi(x)) = 0$ ■

Why not use Pauli eq.?

γ^2 not time dependent?

$A_\mu^* = A_\mu$? $\partial_\mu^* = \partial_\mu$.

Can you just multiply an equation by a matrix? γ^2

c) $\psi(t,\vec x) = e^{-iEt}\phi(\vec x)$ Stationary state: $\frac{\partial}{\partial t}\|\psi(t,\vec x)\|^2 = \partial_t\|\phi(\vec x)\|^2 = 0$

$\underbrace{i\frac{\partial}{\partial t}}_{\hat E}\psi(t,\vec x) = i(-iE)\psi(t,\vec x) = E\psi(t,\vec x)$ with energy E

Now: $\hat C\psi = i\gamma^2\psi^*(t,\vec x) = i\gamma^2 e^{iEt}\phi^*(\vec x) = e^{iEt}\tilde\phi(\vec x)$, $\tilde\phi(\vec x) := i\gamma^2\phi^*(\vec x)$

$\rightsquigarrow \partial_t\|\hat C\psi\|^2 = \partial_t\|\tilde\phi(\vec x)\|^2 = 0$ Stationary

and $i\frac{\partial}{\partial t}(\hat C\psi(t,\vec x)) = i(iE)i\gamma^2 e^{iEt}\phi^*(\vec x) = -E(i\gamma^2 e^{iEt}\phi^*(\vec x))$

$= -E(\hat C\psi(t,\vec x))$

How to see that $\hat C\psi$ is a stationary state again?

$\phi^*(\vec x)$?

$\phi(\vec x) \in \mathbb{C}^4$ again?

What norm for spinors?