Advanced theoretical Particle Physics Homework 11

07.07.2018 1) Had for the scalar potential

$$\mathcal{L} \supset V = \sum_i \left| \frac{\partial W}{\partial \Phi_i}\Big|_{\theta=0} \right|^2 = \sum_i W_i \bar{W}^i = \sum_i |F_i|^2$$

✓

where $W_i(\phi) = \frac{\partial W}{\partial \Phi_i}\Big|$, $\bar{W}^i(\phi^*) = \frac{\partial W^\dagger}{\partial \Phi_i^\dagger}$

What exactly $\bar{W}^{ij}(\phi^*)$? Analogue as for $\bar{W}^i$? ⇝ yes, see book

Also $W_{ij}(\phi) = \frac{\partial^2 W}{\partial \Phi_i \partial \Phi_j}\Big|$, $\bar{W}^{ij}(\phi^*) = \frac{\partial^2 W^\dagger}{\partial \Phi_i^\dagger \partial \Phi_j^\dagger}$

$W_{ijk}(\phi) = \frac{\partial^3 W}{\partial \Phi_i \partial \Phi_j \partial \Phi_k}$, $\bar{W}^{ijk}(\phi^*) = \frac{\partial^3 W^\dagger}{\partial \Phi_i^\dagger \partial \Phi_j^\dagger \partial \Phi_k^\dagger}$

and we used the eq. of constraints $F_i = -\bar{W}^i$

$F_i^* = -W_i$

Rather defined than found this for $m^{(f)}$?

For the fermionic mass matrix, one defines $m_{ij}^{(f)} = \langle W_{ij} \rangle \equiv W_{ij}(\langle\phi$

¶ the scalar mass term in $\mathcal{L}$ takes the form

✓

$$\mathcal{L}_{SMT} = -\frac{1}{2}(\varphi_i, \varphi_i^*)\, m_{ij}^{(s)2} \begin{pmatrix} \varphi_j \\ \varphi_j^* \end{pmatrix} \leftarrow \text{rather} \begin{pmatrix} \varphi_j^* \\ \varphi_j \end{pmatrix} ?$$

Where is the 1/2 in the mass term comming from? ⇝ just different param. (def.; e.g. $-\frac{1}{2}\varphi_i\varphi_j m_{ij}$ Product rule cancels the 1/2

the general form of the superpotential is

$$W(\Phi_\ell) = l_\ell \Phi_\ell + \frac{1}{2} m_{\ell m}\Phi_\ell \Phi_m + \frac{1}{6} f_{\ell m n}\Phi_\ell \Phi_m \Phi_n$$

One then finds:

m, f totally sym.

$$W_i(\phi) \overset{\swarrow}{=} l_i + m_{i\ell}\phi_\ell + \frac{1}{2} f_{imn}\phi_m \phi_n$$

$$\bar{W}^i(\phi^*) = l_i^* + m_{i\ell}^*\phi_\ell^* + \frac{1}{2} f_{imn}^*\phi_m^* \phi_n^*$$

$$W_{ij}(\phi) = m_{ij} + f_{ij\ell}\phi_\ell$$

$$\bar{W}^{ij}(\phi^*) = m_{ij}^* + f_{ij\ell}^*\phi_\ell^*$$

✓

$$W_{ijk}(\phi) = f_{ijk} = \left(\bar{W}^{ijk}(\phi^*)\right)^*$$

Can this be exp. further? ⇝ not needed

$$\rightsquigarrow V = \sum_i \left\{ l_i + m_{i\ell}\phi_\ell + \frac{1}{2} f_{imn}\phi_m\phi_n \right\}\left\{ l_i^* + m_{ia}^*\phi_a^* + \frac{1}{2} f_{iab}^*\phi_a^*\phi_b^* \right\}$$

The mass terms can then be obtained by taking subsequent derivatives w.r.t. $\phi_x, \phi_y^*\ldots$

where $m^{(S)2}_{ij}$ takes the form of a 2x2 matrix

$$m^{(S)2}_{ij} = \begin{pmatrix} (m^{(S)2}_{ij})_{11} & (m^{(S)2}_{ij})_{12} \\ (m^{(S)2}_{ij})_{21} & (m^{(S)2}_{ij})_{22} \end{pmatrix} = \begin{pmatrix} \frac{\partial V}{\partial\phi_i\partial\phi_j^*} & \frac{\partial V}{\partial\phi_i\partial\phi_j} \\ \frac{\partial V}{\partial\phi_i^*\partial\phi_j^*} & \frac{\partial V}{\partial\phi_i^*\partial\phi_j} \end{pmatrix}$$

$\phi = \langle\phi\rangle$

and one easily verifies

$$(m^{(S)2}_{ij})_{11} = \frac{\partial^2 V}{\partial\phi_i\,\partial\phi_j^*}\Big|, \quad (m^{(S)2}_{ij})_{12} = \frac{\partial^2 V}{\partial\phi_i\partial\phi_j}\Big|$$

$$(m^{(S)2}_{ij})_{21} = \frac{\partial^2 V}{\partial\phi_i^*\,\partial\phi_j^*}\Big|, \quad (m^{(S)2}_{ij})_{22} = \frac{\partial^2 V}{\partial\phi_i^*\partial\phi_j}\Big|$$

✓ Sym in i,j but the 2x2 matrix isn't sym? ⇝ not necessarily as m, f can be compl

$$\leadsto (m^{(S)2}_{ij})_{11} = \sum_p \{m_{pi} + f_{pin}\phi_n\}\{m_{pj}^* + f_{pjb}^*\phi_b^*\}\Big|_{\phi=\langle\phi\rangle} = \sum_p W_{pi}(\langle\phi\rangle)\overline{W}^{pj}(\langle\phi\rangle)$$

$$(m^{(S)2}_{ij})_{12} = \sum_p \{f_{pij}\}\{l_p^* + m_{pa}^*\phi_a^* + \tfrac{1}{2}f_{pab}^*\phi_a^*\phi_b^*\}\Big| = \sum_p W_{pij}(\langle\phi\rangle)\overline{W}^p(\langle\phi\rangle)$$

✓

$$(m^{(S)2}_{ij})_{21} = \sum_p \{l_p + m_{pe}\phi_e + \tfrac{1}{2}f_{pmn}\phi_m\phi_n\}f_{pij}\Big| = \sum_p W_p(\langle\phi\rangle)\overline{W}^{pij}(\langle\phi\rangle) \quad \overline{W}^{pij} = W_{pij}?$$

No ϕ, ϕ^* dep.? ⇝ for real coeff.

$$(m^{(S)2}_{ij})_{22} = \sum_p \{m_{pj} + f_{pjn}\phi_n\}\{m_{pi}^* + f_{pib}^*\phi_b^*\}\Big| = \sum_p W_{pj}(\langle\phi\rangle)\overline{W}^{pi}(\langle\phi\rangle)$$

b) We then find for $\mathrm{tr}\, m^{(s)2}$ that

✓

Why also trace over the i j indices?
$\rightsquigarrow$ e.g. $\langle W_{ki} \rangle \langle \overline{W}^{kj} \rangle$ again N×N matrices and we have to sum it's diag. entries

$$\mathrm{tr}\left(m^{(s)2}\right) = \left(m_{ii}^{(s)2}\right)_{11} + \left(m_{ii}^{(s)2}\right)_{22}$$

$$= \sum_p W_{pi}(\langle\phi\rangle)\, \overline{W}^{pi}(\langle\phi\rangle) + \sum_p W_{pi}(\langle\phi\rangle)\, \overline{W}^{pi}(\langle\phi\rangle)$$

$$\left| \; m_{ij}^{(f)} = \langle W_{ij} \rangle = W_{ij}(\langle\phi\rangle_k) \right.$$

$$\rightsquigarrow m_{ij}^{*(f)} = \left(W_{ij}(\langle\phi\rangle_k)\right)^* = W_{ij}(\langle\phi\rangle_k^*)$$

$$\left| \; \begin{aligned} &= \overline{W}^{ij}(\langle\phi\rangle_k) \\ &= m^{\dagger}{}_{ji}^{(f)} = m^{\dagger}{}_{ij}^{(f)} \end{aligned} \right.$$

First the relation w/ $mm^\dagger + m^\dagger m$?

$$= \sum_p m_{pi}^{(f)} m_{pi}^{(f)*} + \sum_p m_{pi}^{(f)*} m_{pi}^{(f)}$$

$$= \sum_p m_{pi}^{(f)} m_{ip}^{(f)\dagger} + \sum_p m_{pi}^{(f)} m_{ip}^{(f)\dagger} = 2\,\mathrm{tr}\left(m^{(f)} m^{(f)\dagger}\right)$$

$$= \mathrm{tr}\left(m^{(f)} m^{(f)\dagger} + m^{(f)\dagger} m^{(f)}\right)$$

✓

But m sym. anyway?
$\rightsquigarrow$ yes, here actually $\mathbb{1}^{(*)} = \mathbb{1}^\dagger$

c) Instead of the bases above, $\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_1^* \\ \varphi_2 \\ \varphi_2^* \\ \vdots \end{pmatrix} \Big\downarrow 2N$, we will now choose $\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_1^* \\ \varphi_2^* \\ \vdots \end{pmatrix} \Big\downarrow 2N$ s.t.

the matrix

$$m^{(s)2} = \overset{2\times 2}{\begin{pmatrix} \left(m_{11}^{(s)2}\right) & \left(m_{12}^{(s)2}\right) & \cdots \\ \left(m_{21}^{(s)2}\right) & \left(m_{22}^{(s)2}\right) & \cdots \\ \vdots & & \end{pmatrix}} \Big\downarrow N$$

$$\xrightarrow{\hspace{6cm}}$$

takes the form

$$m^{(s)2} = \overset{N}{\begin{pmatrix} \left(m^{(s)2}\right)_{11} & \left(m^{(s)2}\right)_{12} \\ \left(m^{(s)2}\right)_{21} & \left(m^{(s)2}\right)_{22} \end{pmatrix}} \underset{N \times N}{\uparrow}$$

With $\left(m^{(s)2}\right)_{ab}$ being $\left(m_{ij}^{(s)2}\right)_{ab}$ for $i,j = 1,\dots,N$

which simply corresponds to a reordering (trafo) of the coordinates.

then with $\langle F_i^* \rangle = -\langle W_i \rangle = 0$, we find

$$(m_{ij}^{(s)2})_{12} = (m_{ij}^{(s)2})_{21} = 0 \quad \text{as} \quad \overline{W}^i(\phi) = W_i(\phi)^*$$

$$\rightsquigarrow m^{(s)2} = \begin{pmatrix} (m^{(s)2})_{11} & 0 \\ 0 & (m^{(s)2})_{22} \end{pmatrix}$$

$$(m_{ij}^{(s)2})_{11} = \sum_p W_{pi}(\langle\phi\rangle)\, \overline{W}^{pj}(\langle\phi\rangle) \overset{\checkmark \text{def}^n \text{ sum conv.}}{=} m_{pi}^{(f)}\, m_{pj}^{(f)*} \overset{m \text{ sym.}}{=} (m^{(f)} m^{(f)\dagger})_{ij}$$

$$(m_{ij}^{(s)2})_{22} = \sum_p W_{pj}(\langle\phi\rangle)\, \overline{W}^{pi}(\langle\phi\rangle) = m_{pj}^{(f)}\, m_{pi}^{(f)*} = (m^{(f)} m^{(f)\dagger})_{ji}$$

where we used $\overline{W}^{ij}(\langle\phi\rangle) = m_{ij}^{(f)*}$.

Also $(m^{(f)} m^{(f)\dagger})_{ji} = ((m^{(f)} m^{(f)\dagger})^T)_{ij} \left(= (m^{(f)\dagger} m^{(f)})_{ij}\right)$

$$\rightsquigarrow m^{(s)2} = \begin{pmatrix} m^{(f)} m^{(f)\dagger} & 0 \\ 0 & (m^{(f)} m^{(f)\dagger})^T \end{pmatrix}$$

✓

Other way around in book?

$\rightsquigarrow \left[\frac{\partial V}{\partial\phi_i \partial\phi_j} \;\; \begin{matrix} 2 \\ 3 \;\; 4 \end{matrix} \right|$ other way around

✓

But $m^{(f)}$ should be real as well? or v.v. complex?

$\rightsquigarrow$ Can be complex!

e.g. $M_{ij}\phi_i\phi_j \rightsquigarrow m_{ij}^* \phi_i^* \phi_j^*$ c.c. masses for c.c. fields

then we find for the char. polynomial:

$$\Lambda := \det(m^{(s)2} - \lambda \mathbb{1}) = \det(m^{(f)} m^{(f)\dagger} - \lambda \mathbb{1}) \det((m^{(f)} m^{(f)\dagger})^T - \lambda \mathbb{1})$$

$$\underset{\det A^T = \det A}{\overset{\wedge}{=}} \det(m^{(f)} m^{(f)\dagger} - \lambda \mathbb{1}) \det(m^{(f)} m^{(f)\dagger} - \lambda \mathbb{1}) = \det(m^{(f)} m^{(f)\dagger} - \lambda \mathbb{1})^2$$

which tells us that we have the same eigenvalues but each with multiplicity 2.

2) One single left chiral superfield Φ with superpotential:

$$W = \frac{1}{2} m \Phi^2 + \frac{1}{6} f \Phi^3$$

where in class we had the general form

$$W(\Phi_i) = l_i \Phi_i + \frac{1}{2} m_{ij} \Phi_i \Phi_j + \frac{1}{3!} f_{ijk} \Phi_i \Phi_j \Phi_k$$

and $$\mathcal{L} = \int d^4\theta \sum_i \Phi_i^\dagger \Phi_i + \left[\int d^2\theta \, W(\Phi_i) + h.c. \right]$$

a)

Has to be written down in x-space as $\Phi_i^\dagger \bar{\Phi}_i$ not left-ch. anymore?

For this one left chiral field, we thus get

$$\mathcal{L} = \int d^4\theta \, \Phi^\dagger \Phi + \left[\int d^2\theta \, W(\Phi) + h.c. \right]$$

We also had

Is it $[\partial_\mu]$-der. for the ϕ_i terms as well? Or $(\partial_\mu \phi_i^*)(\partial^\mu \phi_k)$ as in lecture? → both correct, differ by a tot. der.

$$\left[\Phi_i^\dagger \Phi_k \right]_{D \hat{=} \theta\theta\bar{\theta}\bar{\theta}} = F_i^*(x) F_k(x) + \frac{1}{2} (\partial_\mu \phi_i^*(x)) [\partial^\mu] \phi_k - \frac{1}{2} \phi_i^*(x) [\partial_\mu] (\partial^\mu \phi_k(x)) + i \xi_k(x) \sigma^\mu [\partial_\mu] \bar{\xi}_i(x)$$

$$\left[\Phi_1 \Phi_2 \right]_{F \hat{=} \theta\theta} = \phi_1(x) F_2(x) + \phi_2(x) F_1(x) - \xi_1 \xi_2$$

$$\left[\Phi_1 \Phi_2 \Phi_3 \right]_{F \hat{=} \theta\theta} = F_1 \phi_2 \phi_3 + F_2 \phi_1 \phi_3 + F_3 \phi_1 \phi_2 - \xi_1 \xi_2 \phi_3 - \xi_1 \xi_3 \phi_2 - \xi_2 \xi_3 \phi_1$$

and thus find:

$$\mathcal{L} = F^*(x) F(x) + \frac{1}{2} (\partial_\mu \phi^*(x)) [\partial^\mu] \phi(x) - \frac{1}{2} \phi^*(x) [\partial_\mu] (\partial^\mu \phi(x)) + i \xi(x) \sigma^\mu [\partial_\mu] \bar{\xi}(x)$$

M and f real as well (+sym.)? → in general can be complex; here need to be assumed real, so can write as Dirac spinor

$$+ \frac{1}{2} m \left\{ \phi(x) F(x) + \phi(x) F(x) - \xi\xi \right\} + \frac{1}{6} f \left\{ F\phi\phi + F\phi\phi + F\phi\phi - \xi\xi\phi - \xi\xi\phi - \xi\xi\phi \right\}$$

$$+ \frac{1}{2} m \left\{ \phi^*(x) F^*(x) + \phi^*(x) F^*(x) - \bar{\xi}\bar{\xi} \right\} + \frac{1}{6} f \left\{ F^*\phi^*\phi^* + F^*\phi^*\phi^* + F^*\phi^*\phi^* - \bar{\xi}\bar{\xi}\phi^* - \bar{\xi}\bar{\xi}\phi^* - \bar{\xi}\bar{\xi}\phi^* \right\}$$

$$= F^*(x)F(x) + \frac{1}{2}\partial_\mu\phi^*(x)[\partial^\mu]\phi(x) - \frac{1}{2}\phi^*(x)[\partial_\mu]\partial^\mu\phi(x)$$

$$+ i\xi(x)\sigma^\mu[\partial_\mu]\bar{\xi}(x)$$

$$+ m\left\{\phi(x)F(x) - \frac{1}{2}\xi\xi\right\} + \frac{1}{2}f\left\{F(x)\phi(x)\phi(x) - \xi(x)\xi(x)\phi(x)\right\}$$

$$+ m\left\{\phi^*(x)F^*(x) - \frac{1}{2}\bar{\xi}\bar{\xi}\right\} + \frac{1}{2}f\left\{F^*(x)\phi^*(x)\phi^*(x) - \bar{\xi}(x)\bar{\xi}(x)\phi^*(x)\right\}$$

b) Have $\frac{\partial \mathcal{L}}{\partial F^*} = 0$ and $\frac{\partial \mathcal{L}}{\partial F} = 0$ as F and F^* don't propagate

$$0 = \frac{\partial \mathcal{L}}{\partial F^*} = F(x) + m\phi^*(x) + \frac{1}{2}f\phi^*(x)\phi^*(x) \rightsquigarrow F(x) = -m\phi^*(x) - \frac{1}{2}f\phi^*(x)\phi^*(x)$$

$$F^*(x) = -m\phi(x) - \frac{1}{2}f\phi(x)\phi(x)$$

analogously

$$\rightsquigarrow \mathcal{L} = \left\{m\phi(x) + \frac{1}{2}f\phi(x)\phi(x)\right\}\left\{m\phi^*(x) + \frac{1}{2}f\phi^*(x)\phi^*(x)\right\}$$

$$+ \frac{1}{2}\partial_\mu\phi^*(x)[\partial^\mu]\phi(x) - \frac{1}{2}\phi^*(x)[\partial_\mu]\partial^\mu\phi(x) + i\xi(x)\sigma^\mu[\partial_\mu]\bar{\xi}(x)$$

$$- m\phi(x)\left\{m\phi^*(x) + \frac{1}{2}f\phi^*(x)\phi^*(x)\right\} - \frac{m}{2}\xi(x)\xi(x)$$

$$- \frac{f}{2}\phi(x)\phi(x)\left\{m\phi^*(x) + \frac{1}{2}f\phi^*(x)\phi^*(x)\right\} - \frac{f}{2}\xi(x)\xi(x)\phi(x)$$

$$- m\phi^*(x)\left\{m\phi(x) + \frac{1}{2}f\phi(x)\phi(x)\right\} - \frac{m}{2}\bar{\xi}(x)\bar{\xi}(x)$$

$$- \frac{f}{2}\phi^*(x)\phi^*(x)\left\{m\phi(x) + \frac{1}{2}f\phi(x)\phi(x)\right\} - \frac{f}{2}\bar{\xi}(x)\bar{\xi}(x)\phi^*(x)$$

$$= \frac{1}{2}\partial_\mu\phi^*(x)[\partial^\mu]\phi(x) - \frac{1}{2}\phi^*(x)[\partial_\mu]\partial^\mu\phi(x) \underbrace{+ i\xi(x)\sigma^\mu[\partial_\mu]\bar{\xi}(x)}_{(1)}$$

$$- m^2\phi^*(x)\phi(x) - \frac{1}{4}f^2\phi(x)\phi(x)\phi^*(x)\phi^*(x)$$

$$- \frac{mf}{2}\phi^*(x)\phi(x)\phi(x) - \frac{mf}{2}\phi(x)\phi^*(x)\phi^*(x)$$

$$\underbrace{- \frac{m}{2}\xi(x)\xi(x) - \frac{m}{2}\bar{\xi}(x)\bar{\xi}(x)}_{(2)} \underbrace{- \frac{f}{2}\xi(x)\xi(x)\phi(x) - \frac{f}{2}\bar{\xi}(x)\bar{\xi}(x)\phi^*(x)}_{(3)}$$

$$(x)\ \frac{1}{2}\phi^*\partial^2\phi\ ?$$

$$= \partial_\mu\phi\,\partial^\mu\phi^*$$

c) Now we will translate this into 4-comp. spinors, i.e. use a Majorana spinor Ψ_M with the Weyl spinor ξ and its conjugate

$$\Psi_M = \begin{pmatrix} \xi_A \\ \bar\xi^{\dot A} \end{pmatrix}, \quad \bar\Psi_M = (\xi^A, \bar\xi_{\dot A})$$

$$\hat= \Psi_M = \begin{pmatrix} \xi_v \\ \xi^{\wedge T} \end{pmatrix} \qquad \bar\Psi_M = (\xi^{\wedge +}, \bar\xi_v)$$

then: $(1) = i\xi(x)\sigma^\mu[\partial_\mu]\bar\xi(x) = \frac{i}{2}\xi(x)\sigma^\mu\partial_\mu\bar\xi(x) - \frac{i}{2}\partial_\mu\xi(x)\sigma^\mu\bar\xi(x)$

$$= \frac{i}{2}\{\xi(x)\sigma^\mu\partial_\mu\bar\xi(x) - \partial_\mu\xi(x)\sigma^\mu\bar\xi(x)\}$$

$$= \frac{i}{2}\{\xi(x)\sigma^\mu\partial_\mu\bar\xi(x) + \bar\xi(x)\bar\sigma^\mu\partial_\mu\xi(x)\}$$

$$\overset{(I.13c)}{=} \frac{i}{2}\{\bar\Psi_M\gamma^\mu\partial_\mu\Psi_M\}$$

✓ Derivative on one of the ξ ⇝ no actual new field? Otherwise identity not possible? Or w/ $\Psi_{\partial M} = \partial_\mu\Psi_M$? ⇝ new field, define $\Psi_{\partial M} = \partial_\mu\Psi_M$

$$(2) = -\frac{m}{2}\{\overset{m}{}\xi(x)\xi(x) + \overset{m^*}{}\bar\xi(x)\bar\xi(x)\}$$

$$\overset{(I.13a)}{=} -\frac{m}{2}\{\bar\Psi_M\Psi_M\}$$

$$(3) = -\frac{f}{2}\{\xi(x)\xi(x)\phi(x) + \bar\xi(x)\bar\xi(x)\phi^*(x)\}$$

$(I.13a) + (I.13b) \rightsquigarrow 2\bar\lambda_1\bar\lambda_2 = \bar\lambda_{1M}(\mathbb{1}+\gamma_5)\lambda_{2M}$

$\rightsquigarrow \bar\lambda_1\bar\lambda_2 = \bar\lambda_{1M}P_R\lambda_{2M}$

$(I.13a) - (I.13b) \rightsquigarrow \lambda_1\lambda_2 = \bar\lambda_{1M}P_L\lambda_{2M}$

$$= -\frac{f}{2}\{\bar\Psi_M P_L\Psi_M\phi(x) + \bar\Psi_M P_R\Psi_M\phi^*(x)\}$$

$P_{L|R}^\dagger = P_{L|R}$, $P_{L|R}^2 = P_{L|R}$, $\Psi^\dagger\gamma^0 P_{L|R} = \bar\Psi_{R|L}$

$$= -\frac{f}{2}\{\bar\Psi_{MR}\Psi_{ML}\phi(x) + \bar\Psi_{ML}\Psi_{MR}\phi^*(x)\}$$

$$= -\frac{f}{2}\{\bar\Psi_{MR}\Psi_{ML}\phi(x) + h.c.\}$$