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https://www.physics-and-stuff.com/

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Theoretical Particle Physics 2 Homework 2 Marvin Zanke

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1) $\frac{d}{dt}|\nu_\alpha\rangle = -iH_{\alpha\beta}|\nu_\beta\rangle$

"heuristically" $\hat{=}$ no quantum field theory!

$|\nu_k\rangle = U_{k\alpha}|\nu_\alpha\rangle$

decoherence? → different masses → different speed → ignore spatial diff here.

$|\nu_k\rangle_t = e^{-iE_k t}|\nu_k\rangle, \quad E_k = \sqrt{m_k^2 + p^2}$

Why do the mass eigenstates evolve in time?

$$E_k = \sqrt{m_k^2+p^2} = p\sqrt{1+\frac{m_k^2}{p^2}} = p\left(1+\frac{m_k^2}{2p^2}+\mathcal{O}\left(\frac{m_k^4}{p^4}\right)\right)$$
$$= p + \frac{1}{2}\frac{m_k^2}{p} + \mathcal{O}\left(\frac{m_k^4}{p^3}\right)$$

Why does U also diagonalize the mass matrix for neutrinos?

For convenience, we use $\psi_m = \begin{pmatrix}\nu_1\\ \nu_2\end{pmatrix}_t, \quad \psi_m = U\psi_f$

$\psi_f = \begin{pmatrix}\nu_e\\ \nu_\mu\end{pmatrix}_t, \quad H = H_f$

w.r.t. diag. the H_f to H_m where H_f is still called a mass matrix with the flavor states. In diagonal form of mass matrix, the mass eigenstates propagate freely in time (diag. matrix)

$\rightsquigarrow \frac{d}{dt}\psi_f = -iH_f\psi_f \quad | \times U$

$\Longleftrightarrow \frac{d}{dt}U\psi_f = -iUH_fU^\dagger U\psi_f$

$\Longleftrightarrow \frac{d}{dt}\psi_m = -iH_m\psi_m, \quad H_m = UH_fU^\dagger$

$\Longleftrightarrow -i\begin{pmatrix}E_1 & 0\\ 0 & E_2\end{pmatrix}\psi_m = -iH_m\psi_m$

$$\rightsquigarrow H_m = \begin{pmatrix}E_1 & 0\\ 0 & E_2\end{pmatrix} = \begin{pmatrix}p+\frac{1}{2}\frac{m_1^2}{p} & 0\\ 0 & p+\frac{1}{2}\frac{m_2^2}{p}\end{pmatrix}$$
$$= p\mathbb{1} + \frac{1}{2p}\begin{pmatrix}m_1^2 & 0\\ 0 & m_2^2\end{pmatrix}$$

2. $U = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$

why neglect the first leading term now? → $|\nu_1\rangle \exp(i\varphi_1)\exp(\text{const}) + |\nu_2\rangle \exp(i\varphi_2)\exp(\text{const})$ because $p\mathbb{1}$ is the same for both neutrinos. And we are only interested in relative phase

$H_f = U^\dagger \tilde{H}_m U, \quad \tilde{H}_m = \frac{1}{2p}\begin{pmatrix}m_1^2 & 0\\ 0 & m_2^2\end{pmatrix}$

$$= \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\frac{1}{2p}\begin{pmatrix}m_1^2 & 0\\ 0 & m_2^2\end{pmatrix}\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$$
$$= \frac{1}{2p}\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}m_1^2\cos\theta & -m_1^2\sin\theta\\ m_2^2\sin\theta & m_2^2\cos\theta\end{pmatrix}$$
$$= \frac{1}{2p}\begin{pmatrix}m_1^2\cos^2\theta + m_2^2\sin^2\theta & -m_1^2\cos\theta\sin\theta + m_2^2\sin\theta\cos\theta\\ -m_1^2\sin\theta\cos\theta + m_2^2\sin\theta\cos\theta & m_1^2\sin^2\theta + m_2^2\cos^2\theta\end{pmatrix}$$

Using $\sin 2\vartheta = \sin\vartheta\cos\vartheta + \sin\vartheta\cos\vartheta$, we find

$$= \frac{1}{2p}\begin{pmatrix} m_1^2\cos^2\vartheta + m_2^2\sin^2\vartheta & \frac{m_2^2-m_1^2}{2}\sin 2\vartheta \\ \frac{m_2^2-m_1^2}{2}\sin 2\vartheta & m_1^2\sin^2\vartheta + m_2^2\cos^2\vartheta \end{pmatrix}$$

$$m_1^2\cos^2\vartheta + m_2^2\sin^2\vartheta = \frac{1}{2}\left(2m_1^2\cos^2\vartheta + 2m_2^2\sin^2\vartheta\right)$$

$$= \frac{1}{2}\left(m_1^2\cos^2\vartheta + m_1^2 - m_1^2\sin^2\vartheta + m_2^2\sin^2\vartheta + m_2^2 - m_2^2\cos^2\vartheta\right)$$

$$= \frac{1}{2}\left(m_1^2 + m_2^2 + m_1^2(\cos^2\vartheta - \sin^2\vartheta) + m_2^2(\sin^2\vartheta - \cos^2\vartheta)\right)$$

$$\overset{\delta m^2 = m_2^2 - m_1^2}{=} \frac{1}{2}\left(m_1^2 + m_2^2 - \delta m^2\cos 2\vartheta\right)$$

$$m_2^2\cos^2\vartheta + m_1^2\sin^2\vartheta = \frac{1}{2}\left(m_1^2 + m_2^2 + m_2^2\cos^2\vartheta - m_2^2\sin^2\vartheta + m_1^2\sin^2\vartheta - m_1^2\cos^2\vartheta\right)$$

$$= \frac{1}{2}\left(m_1^2 + m_2^2 + \delta m^2\cos 2\vartheta\right)$$

$$\overset{\delta m^2 = m_2^2 - m_1^2}{=} \frac{1}{2p}\left(\frac{m_1^2+m_2^2}{2}\mathbb{1} + \frac{\delta m^2}{2}\begin{pmatrix} -\cos 2\vartheta & \sin 2\vartheta \\ \sin 2\vartheta & \cos 2\vartheta \end{pmatrix}\right)$$

✓

Again neglect first term — why?
⇝ Same argument, if it was in a medium it would be more difficult as extra [illegible] terms appear

3. $U = \begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix}$, $\tilde{\tilde{H}}_f = \frac{\delta m^2}{4p}\begin{pmatrix} -\cos 2\vartheta & \sin 2\vartheta \\ \sin 2\vartheta & \cos 2\vartheta \end{pmatrix}$

$a_1 = \begin{pmatrix} \cos\vartheta \\ -\sin\vartheta \end{pmatrix}$, $a_2 = \begin{pmatrix} \sin\vartheta \\ \cos\vartheta \end{pmatrix}$

$$\tilde{\tilde{H}}_f a_1 = \frac{\delta m^2}{4p}\begin{pmatrix} -\cos 2\vartheta & \sin 2\vartheta \\ \sin 2\vartheta & \cos 2\vartheta \end{pmatrix}\begin{pmatrix} \cos\vartheta \\ -\sin\vartheta \end{pmatrix} = \frac{\delta m^2}{4p}\begin{pmatrix} -\cos\vartheta\cos 2\vartheta - \sin\vartheta\sin 2\vartheta \\ \sin 2\vartheta\cos\vartheta - \cos 2\vartheta\sin\vartheta \end{pmatrix}$$

$$= \frac{\delta m^2}{4p}\begin{pmatrix} -\cos\vartheta(\cos^2\vartheta - \sin^2\vartheta) - \sin\vartheta(\sin\vartheta\cos\vartheta + \sin\vartheta\cos\vartheta) \\ \cos\vartheta(\sin\vartheta\cos\vartheta + \sin\vartheta\cos\vartheta) - \sin\vartheta(\cos^2\vartheta - \sin^2\vartheta) \end{pmatrix}$$

$$= \frac{\delta m^2}{4p}\begin{pmatrix} -\cos^3\vartheta + \sin^2\vartheta\cos\vartheta - \sin^2\vartheta\cos\vartheta - \sin^2\vartheta\cos\vartheta \\ \sin\vartheta\cos^2\vartheta + \sin\vartheta\cos^2\vartheta - \sin\vartheta\cos^2\vartheta + \sin^3\vartheta \end{pmatrix} = \frac{\delta m^2}{4p}\begin{pmatrix} -\cos\vartheta \\ \sin\vartheta \end{pmatrix} = -\frac{\delta m^2}{4p}\begin{pmatrix} \cos\vartheta \\ -\sin\vartheta \end{pmatrix}$$

$$\tilde{\tilde{H}}_f a_2 = \frac{\delta m^2}{4p}\begin{pmatrix} -\cos 2\vartheta & \sin 2\vartheta \\ \sin 2\vartheta & \cos 2\vartheta \end{pmatrix}\begin{pmatrix} \sin\vartheta \\ \cos\vartheta \end{pmatrix}$$

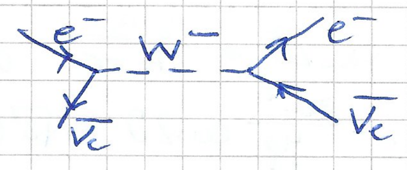
$$= \frac{\delta m^2}{4p}\begin{pmatrix} -\sin\vartheta(\cos^2\vartheta - \sin^2\vartheta) + \cos\vartheta(\sin\vartheta\cos\vartheta + \sin\vartheta\cos\vartheta) \\ \sin\vartheta(\sin\vartheta\cos\vartheta + \sin\vartheta\cos\vartheta) + \cos\vartheta(\cos^2\vartheta - \sin^2\vartheta) \end{pmatrix}$$

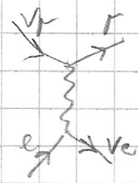
Solution of (1) then proceeds as in class?

$$= \frac{\delta m^2}{4p}\begin{pmatrix} -\sin\vartheta\cos^2\vartheta + \sin^3\vartheta + \sin\vartheta\cos^2\vartheta + \sin\vartheta\cos^2\vartheta \\ \sin^2\vartheta\cos\vartheta + \sin^2\vartheta\cos\vartheta + \cos^3\vartheta - \cos\vartheta\sin^2\vartheta \end{pmatrix} = \frac{\delta m^2}{4p}\begin{pmatrix} \sin\vartheta \\ \cos\vartheta \end{pmatrix}$$

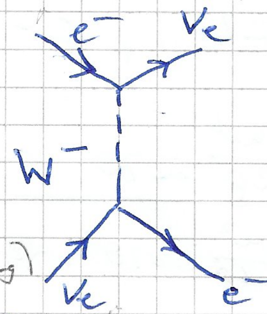
4. $e^- + \bar{\nu}_e \to e^- + \bar{\nu}_e$

$e^- + \nu_e \to e^- + \nu_e$





not to forward scattering (state in and state out) as ν changes and also energy / momentum difficult w/ μ in final state (for forward scattering)



why only electron neutrinos? ⇝ other diagrams (no free muons!) don't contribute to forw. scattering

but no „free" muons in the sun.

$$\rightsquigarrow \Delta H = \begin{pmatrix} \sqrt{2}\, G_F n_e & 0 \\ 0 & 0 \end{pmatrix}$$

where from exact form of ΔH? ⇝ (look up (internet)?) "not that hard"

5. $$\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} \cos\tilde{\theta} & \sin\tilde{\theta} \\ -\sin\tilde{\theta} & \cos\tilde{\theta} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$$

2

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix} \text{ form of } H + \Delta H = \frac{\delta m^2}{4p} \begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} + \begin{pmatrix} \sqrt{2}\, G_F n_e & 0 \\ 0 & 0 \end{pmatrix}$$

why use rows and not columns? As $U \neq U^\dagger$ now?

$\rightsquigarrow \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\tilde{\theta} & -\sin\tilde{\theta} \\ \sin\tilde{\theta} & \cos\tilde{\theta} \end{pmatrix}}_{U^\dagger} \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix}$

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix} \begin{pmatrix} \cos\tilde{\theta} \\ \sin\tilde{\theta} \end{pmatrix} = \lambda \begin{pmatrix} \cos\tilde{\theta} \\ \sin\tilde{\theta} \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} A\cos\tilde{\theta} + B\sin\tilde{\theta} \\ B\cos\tilde{\theta} + C\sin\tilde{\theta} \end{pmatrix} = \lambda \begin{pmatrix} \cos\tilde{\theta} \\ \sin\tilde{\theta} \end{pmatrix}$$

$\frac{\partial}{\partial t}\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = (H+\Delta H)\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} \,\Big|\, \times U^\dagger$

$\frac{\partial}{\partial t}\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \underbrace{U^\dagger (H+\Delta H) U}_{= A^{diag}} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$

$A^{diag} = U^\dagger (H+\Delta H) U$ in contrary to $H_m = U H_f U^\dagger$

$$\rightsquigarrow A + B\frac{\sin\tilde{\theta}}{\cos\tilde{\theta}} = \lambda = B\frac{\cos\tilde{\theta}}{\sin\tilde{\theta}} + C$$

$$\Leftrightarrow C - A = B\left(\frac{\sin\tilde{\theta}}{\cos\tilde{\theta}} - \frac{\cos\tilde{\theta}}{\sin\tilde{\theta}}\right)$$

$$\Leftrightarrow C - A = B\left(\frac{\sin^2\tilde{\theta} - \cos^2\tilde{\theta}}{\sin\tilde{\theta}\cos\tilde{\theta}}\right) = B\frac{-\cos 2\tilde{\theta}}{\frac{1}{2}\sin 2\tilde{\theta}} = -2B\frac{1}{\tan 2\tilde{\theta}}$$

$$\Leftrightarrow \tan 2\tilde{\theta} = \frac{2B}{A - C}$$

$$\det\begin{pmatrix} A-\lambda & B \\ B & C-\lambda \end{pmatrix} = (A-\lambda)(C-\lambda) - B^2 \overset{!}{=} 0$$

$$\Leftrightarrow AC + \lambda^2 - \lambda(A+C) - B^2 \overset{!}{=} 0$$

$$\rightsquigarrow \frac{A+C}{2} \pm \sqrt{\frac{(A+C)^2}{4} - AC + B^2} = \frac{A+C}{2} \pm \sqrt{\frac{(A-C)^2}{4} + B^2}$$

So $\tan 2\tilde{\vartheta} = \frac{2B}{A-C}$ and EV: $\frac{A+C}{2} \pm \sqrt{\frac{(A-C)^2}{4} + B^2}$

with $A = -\frac{\delta m^2}{4p}\cos 2\vartheta + \sqrt{2}\, G_F n_e$

$B = \frac{\delta m^2}{4p}\sin 2\vartheta$, $C = \frac{\delta m^2}{4p}\cos 2\vartheta$

$$\Rightarrow \tan 2\tilde{\vartheta} = \frac{\frac{\delta m^2}{2p}\sin 2\vartheta}{-\frac{\delta m^2}{2p}\cos 2\vartheta + \sqrt{2}\, G_F n_e} = \frac{\sin 2\vartheta}{\frac{2\sqrt{2}\, p\, G_F n_e}{\delta m^2} - \cos 2\vartheta}$$

$$\text{EV: } \frac{\sqrt{2}\, G_F n_e}{2} \pm \sqrt{\frac{\left(\sqrt{2}\, G_F n_e - \frac{\delta m^2}{2p}\cos 2\vartheta\right)^2}{4} + \frac{(\delta m^2)^2}{16p^2}\sin^2 2\vartheta}$$

$$\frac{\sqrt{2}\, G_F n_e}{2} \pm \sqrt{\frac{1}{2} G_F^2 n_e^2 + \frac{(\delta m^2)^2}{16p^2} - \frac{\sqrt{2}}{4} G_F n_e \frac{\delta m^2}{p}\cos 2\vartheta}$$

2) $-\mathcal{L}_{mass} = \frac{1}{2}\left[(m\phi + M\Phi)^2 + m^2\Phi^2\right]$

with $m \ll M$, thus $\Phi \approx$ const

$\rightsquigarrow \frac{\partial \mathcal{L}}{\partial \Phi} = 0$

$-\mathcal{L}_{mass} = \frac{1}{2}\left[m^2\phi^2 + M^2\Phi^2 + 2mM\phi\Phi + m^2\Phi^2\right]$

Why only mass-term and get Euler Lagrange eq.
$\rightsquigarrow$ not enough energy/momentum to create heavy $\rightsquigarrow \mathcal{L}$ already w/o $(\partial_\mu \phi)^2$
small

1.

why $\frac{\partial \mathcal{L}}{\partial \Phi} =$ const? And E.L. eq.? $\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)}$?

$\rightsquigarrow$ e.l. eq. with $\mathcal{L} = \mathcal{L}_M$

$\rightsquigarrow 0 = \frac{\partial \mathcal{L}}{\partial \Phi} \Leftrightarrow 0 = -\frac{\partial \mathcal{L}}{\partial \Phi}$

$\rightsquigarrow 0 = M^2\Phi + mM\phi + m^2\Phi = \Phi(M^2+m^2) + mM\phi$

$\Longleftrightarrow \Phi = \frac{-mM}{M^2+m^2}\phi$

Multiply the bracket "out"? $\rightarrow$ only because bosons?! Don't necessarily commute?!
$\rightsquigarrow$ not important if make "unequal" fields commute or anticommute
In Peskin anticommute. Have structure $\bar{\psi}_1 \psi_2$ and have to take care then of $(-)$ signs

$\rightsquigarrow -\mathcal{L}_{mass} = \frac{1}{2}\left[m^2\phi^2 + M^2\frac{m^2M^2}{(M^2+m^2)^2}\phi^2 + 2mM\phi\left(\frac{-mM}{M^2+m^2}\phi\right) + m^2\frac{m^2M^2}{(M^2+m^2)^2}\phi^2\right]$

$= \frac{1}{2}\left[m^2\phi^2 + \frac{m^2M^2}{M^2+m^2}\phi^2 - \frac{2m^2M^2}{M^2+m^2}\phi^2\right]$

$= \frac{1}{2}\left[m^2\phi^2 - \frac{m^2M^2}{M^2+m^2}\phi^2\right]$

$= \frac{1}{2}\phi^2\frac{m^2(M^2+m^2) - m^2M^2}{M^2+m^2} = \frac{1}{2}\phi^2\frac{m^4}{M^2+m^2}$

2. $\mathcal{M}^2_{ij} = \frac{\partial^2 \mathcal{L}}{\partial\phi_i \partial\phi_j} \rightsquigarrow \mathcal{M}^2 = \begin{pmatrix} m^2 & mM \\ mM & M^2+m^2 \end{pmatrix}$

Factor of $(-)$ for masses if $-\mathcal{L}_{mass}$!
$\rightsquigarrow$ Have $\mathcal{L}_M = -\frac{1}{2}(\ )(M)(\ )$ so factor $(-)$ absorbed before

$\rightsquigarrow \begin{vmatrix} m^2-\lambda & mM \\ mM & M^2+m^2-\lambda \end{vmatrix} = (m^2-\lambda)(M^2+m^2-\lambda) - m^2M^2$

$= m^4 - \lambda(2m^2+M^2) + \lambda^2 \stackrel{!}{=} 0$

$\rightsquigarrow \frac{2m^2+M^2}{2} \pm \sqrt{\frac{(2m^2+M^2)^2}{4} - m^4}$

$\rightsquigarrow \frac{2m^2+M^2}{2} \pm \sqrt{\frac{M^4}{4} + m^2M^2}$

$\rightsquigarrow \frac{2m^2+M^2}{2} \pm \frac{M^2}{2}\sqrt{1+4\frac{m^2}{M^2}}$

The smaller eigenvalue yields

$$\frac{2m^2+M^2}{2} - \frac{M^2}{2}\left(1 + 2\frac{m^2}{M^2} - 2\frac{m^4}{M^4} + \mathcal{O}\left(\frac{m^6}{M^6}\right)\right)$$

$$= \frac{2m^2+M^2}{2} - \frac{M^2}{2} - m^2 + \frac{m^4}{M^2} + \mathcal{O}\left(\frac{m^6}{M^4}\right) = \frac{m^4}{M^2} + \mathcal{O}\left(\frac{m^6}{M^4}\right)$$

where we used $\sqrt{1+4x^2} = \sqrt{1+(2x)^2} = 1+2x-2x^2+\dots$

3. For the resumming of the propagator, we denote

$$\overset{\phi}{\text{———}} \mathrel{\hat{=}} \frac{i}{p^2-m^2} \mathrel{\hat{=}} D_\phi$$

$$\overset{\Phi}{═══} \mathrel{\hat{=}} \frac{i}{p^2-M^2-m^2} \mathrel{\hat{=}} D_\Phi$$

$$\text{———}═══ \mathrel{\hat{=}} iMm \mathrel{\hat{=}} C$$

✓ Vertex only factor (i)? No factor (-1)? ⇝ Yes, gets factor i for vertices

Then $D \mathrel{\hat{=}}$ ——— + ———═══——— + ———═══———═══——— + ...

$$= D_\phi + D_\phi C D_\Phi C D_\phi + \dots$$

$$= D_\phi \left\{ \mathbb{1} + C D_\Phi C D_\phi + \dots \right\}$$

$$= D_\phi \sum_{n=0}^{\infty} \left(C D_\Phi C D_\phi\right)^n = \frac{D_\phi}{1 - C D_\Phi C D_\phi}$$

$$= \frac{1}{\frac{1}{D_\phi} + M^2 m^2 D_\Phi} = \frac{1}{\frac{p^2-m^2}{i} + M^2m^2 \frac{i}{p^2-M^2-m^2}}$$

$$= \frac{i}{(p^2-m^2) - \frac{m^2M^2}{p^2-M^2-m^2}}$$

For the pole, we find: $0 = (p^2-m^2) - \frac{m^2M^2}{p^2-M^2-m^2} = \frac{(p^2-m^2)(p^2-M^2-m^2) - m^2M^2}{p^2-M^2-m^2}$

$$\Longleftrightarrow p^4 - p^2M^2 - p^2m^2 - p^2m^2 + m^4 = 0$$

$\xi = p^2$: $\Longleftrightarrow \xi^2 - \xi(M^2+2m^2) + m^4 = 0 \rightsquigarrow \frac{M^2+2m^2}{2} \pm \sqrt{\frac{(M^2+2m^2)^2}{4} - m^4}$

neg. sol. $\rightsquigarrow \frac{M^2+2m^2}{2} - \underbrace{\sqrt{\frac{M^4}{4} + m^2M^2}}_{\frac{M^2}{2}\sqrt{1+4\frac{m^2}{M^2}}} = \frac{M^2+2m^2}{2} - \frac{M^2}{2}\left(1 + 2\frac{m^2}{M^2} - 2\frac{m^4}{M^4} + \dots\right)$ ✓

Has 4 poles now? ⇝ only 2 poles in p^2!

$\rightsquigarrow \frac{m^4}{M^2}$ for the pole.