Disclaimer

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https://www.physics-and-stuff.com/

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Advanced Theoretical Particle Physics Homework 3

05.05.2018

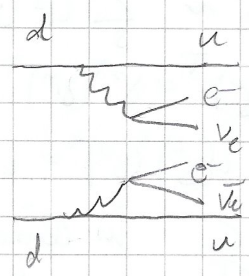
1) Neutrinoless double β decay ($0\nu\beta\beta$)

$$(A,Z) \to (A, Z+2) + e^- + e^-$$

a) Have $2n \to 2p + 2e^-$ and on quark level

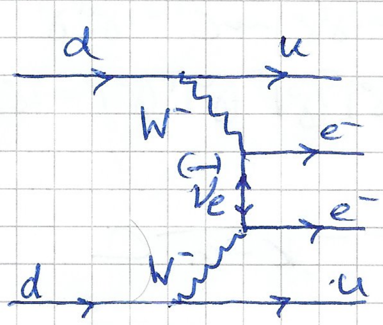
$$2d \to 2u + 2e^-$$

Or better $d \to u + e^- + \bar{\nu}_e$ and sth. w/ the $\bar{\nu}_e$?



$2d \to 2u + 2W^-$
$\hookrightarrow \bar{\nu}_e + e^-$

b)



Why assume non-vanishing neutrinomass? ⇝ Not reality, amplitude vanishes.

Is it $\overset{(-)}{\nu_e}$ or ν_e? Clashing arrows? ⇝ doesn't matter for Maj. Particles

c) The charge conjugation for a field is defined by

$$\psi^c = C\bar{\psi}^T \quad , \quad C = i\gamma^2\gamma^0 \, , \quad C^\dagger = -C \quad ; \quad CC^\dagger = \mathbb{1}$$

$C^2 \neq \mathbb{1}$? $C^{-1} = -C$

We also use $u(k,s) = C\bar{v}^T(k,s)$, $v(k,s) = C\bar{u}^T(k,s)$

$$\bar{v}(k,s) = -u^T(k,s)\,C^{-1} \, , \quad \bar{u}(k,s) = -v^T(k,s)\,C^{-1}$$

C has no effect on a_p^s etc.?

For a fermionic field, the Fourier decomposition reads as follows:

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left(a_p^s u^s(p) e^{-ipx} + b_p^{s\dagger} v^s(p) e^{ipx} \right)$$

right- / left-handed doesn't go in $\psi(x)$?

$$\rightsquigarrow \bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left(a_p^{s\dagger} \bar{u}^s(p) e^{ipx} + b_p^s \bar{v}^s(p) e^{-ipx} \right)$$

$$\rightsquigarrow C\bar{\psi}^T = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left(a_p^{s\dagger} v^s(p) e^{ipx} + b_p^s u^s(p) e^{-ipx} \right)$$

$a_p^{s\,T} = a_p^{s*}$?

Demanding $\psi^c = \psi$ yields $a_p^{s\dagger} = b_p^{s\dagger}$, $a_p^s = b_p^s$

$$\rightsquigarrow \psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left(a_p^s u^s(p) e^{-ipx} + a_p^{s\dagger} v^s(p) e^{ipx} \right)$$

$$\bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left(a_p^{s\dagger} \bar{u}^s(p) e^{ipx} + a_p^s \bar{v}^s(p) e^{-ipx} \right)$$

$$\psi^T(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left(a_p^s u^{sT}(p) e^{-ipx} + a_p^{s\dagger} v^{sT}(p) e^{ipx} \right)$$

$$\bar{\psi} C^T = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left(a_p^{s\dagger} \bar{u}^s(p) C^T e^{ipx} + a_p^s \bar{v}^s(p) C^T e^{-ipx} \right)$$

using $(C\bar{u}^{sT})^T = v^{sT}$ and $(C\bar{v}^{sT})^T = u^{sT}$, we find $\psi^T = \bar{\psi} C^T$.

(arXiv: 1412.3320v1, 10 Dec. 2014, Phys. Review D, volume 45, no. 5, 1 Mar 1992)

We get new Wick contractions of the form $\langle 0 | T[\psi(x)\, \psi^T(y)] | 0 \rangle$

$$= \langle 0 | T[\psi(x)\, \bar{\psi}(y) C^T] | 0 \rangle$$

This yields a contribution $\sim \sum_s u^s \bar{u}^s C^T = -(\not{k} + m) C$

using $(\gamma^\mu)^* = (\gamma^0 \gamma^\mu \gamma^0)^T = \gamma^{0T} (\gamma^\mu)^T \gamma^{0T}$

$$\Rightarrow C^T = -C^* = i(\gamma^2)^* (\gamma^0)^* = i(\gamma^0)^T (\gamma^2)^T (\gamma^0)^T (\gamma^0)^T (\gamma^0)^T (\gamma^0)^T = i\gamma^0\gamma^2 = C^{-1}$$

$$= C^\dagger = -C \quad \text{in chiral basis}$$

$$\Rightarrow \tilde{S}(k) = \frac{i \sum_s u^s \bar{u}^s C^T}{k^2 - m^2} = -\frac{i(\not{k} + m)}{k^2 - m^2} C = -\frac{i}{\not{k} - m} C$$

How to see those contractions ?

why comm. relations and not anti-com. relations ?

What is $[\bar{\psi}, \psi]$ then ? No matrix ? And $T[\dots]$ not the com. but time-ordering w/o com. So not able to use (anti) com. rel. ? But what about $\bar{\psi}(y) C^T |0\rangle$ then ?

2) a) We first check that the given conditions are fulfilled "with our SM".
It's enough to check it for each family separately, as Y and I_3 are the same for the constituents of the family, i.e. use u - for up-like, d - for downlike and l, ν_e for the lepton dublett.

"Chiral fermion"? → left/right handed

We also have to take into account that quarks appear in 3 different colors each.

No counterpart - but could introduce counter-term?

$$Y(u_L) = Y(d_L) = 1/6, \quad Y(u_R) = 2/3, \quad Y(d_R) = -1/3$$

$$Y(l) = Y(\nu_e) = -1/2, \quad Y(e_R) = -1$$

$$I_3(u_L) = 1/2, \quad I_3(d_L) = -1/2, \quad I_3(u_R) = I_3(d_R) = 0$$

$$I_3(l) = -1/2, \quad I_3(\nu_e) = 1/2, \quad I_3(e_R) = 0$$

Why classical L? Actually other way around? Sym of classical L, none in q.m.?

$$\sum_{\text{quarks } q_i} Y(q_i) = 3 \cdot \left\{ 1/6 + 1/6 - 2/3 + 1/3 \right\} = 0$$

using $Y(u_R^c) = -Y(u_R)$, $Y(d_R^c) = -Y(d_R)$

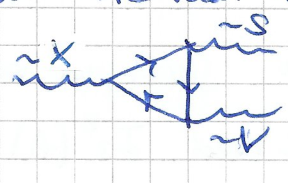
All fermions of given helicity? Why? Only left-handed? Helicity ≠ Chirality? → only l.h.

$$\sum_{\text{fermions } f_i} I_3(f_i)^2 Y(f_i) = 3 \cdot \left\{ \tfrac{1}{4} \cdot \tfrac{1}{6} + \tfrac{1}{4} \cdot \tfrac{1}{6} + 0 \cdot (-\tfrac{2}{3}) + 0 \cdot (\tfrac{1}{3}) \right\} + \tfrac{1}{4}(-\tfrac{1}{2}) + \tfrac{1}{4} \cdot (-\tfrac{1}{2}) + 0 \cdot (1) = 0$$

$$\sum_{\text{fermions } f_i} Y(f_i)^3 = 3 \left\{ (\tfrac{1}{6})^3 + (\tfrac{1}{6})^3 + (-\tfrac{2}{3})^3 + (\tfrac{1}{3})^3 \right\} + (-\tfrac{1}{2})^3 + (-\tfrac{1}{2})^3 + 1^3 = \frac{1}{36} - \frac{8}{9} + \frac{1}{9} - \frac{1}{8} - \frac{1}{8} + 1 = 0$$

I_3 does nothing on charge conj.?

To find the anomaly cancellation requirements, we note that the anomalous part of the triangle diagram



is $\sim tr(\{X, S\} V)$ with X, S, V being the corresponding generators / group matrices for the vertex.

Not a massless theory and thus extra term $m \, Tr \, \gamma_5$ in anomaly?

Let's denote by λ^a the generators of $SU(3)$ and by τ^a the generators of $SU(2)$ while for $U(1)_Y$, we simply have Y.
The product for two different groups is then simply understood as the tensor product with the unit matrix on the other space, i.e.

Vector-like theory → same way under SU(3)? → makes $SU(3) \times SU(3) \times SU(3) = 0$

$$T^a_{SU(3)} = \lambda^a \otimes 1_{SU(2)}, \quad T^a_{SU(2)} = 1_{SU(3)} \otimes \tau^a \quad \text{etc.}$$

Why triangle?

For the trace, we then get $Tr(T^a_{SU(3)} \otimes T^b_{SU(2)}) = Tr(T^a_{SU(3)}) Tr(T^b_{SU(2)})$

- $SU(2)\times SU(2)\times SU(2)$: $Tr(\{\tau^a, \tau^b\}\tau^c) \sim Tr(\tau^c) = 0$ as traceless and $\{\tau^a, \tau^b\} = 2\delta^{ab}\mathbb{1}$
- $SU(2)\times SU(2)\times SU(3)$: $Tr(\{T^a_{SU(2)}, T^b_{SU(2)}\} T^c_{SU(3)})$
  $\sim Tr(2(\mathbb{1}_{SU(3)} \otimes \delta^{ab}\mathbb{1}_{SU(2)})(\lambda^c \otimes \mathbb{1}_{SU(2)})) \sim Tr(\lambda^c) = 0$
- $SU(2)\times SU(3)\times SU(3)$: $Tr(\{T^a_{SU(3)}, T^b_{SU(3)}\} T^c_{SU(2)})$
  $\sim Tr((\frac{1}{3}\delta^{ab} + d^{abd} t^d \otimes 2\mathbb{1}_{SU(2)})(\mathbb{1}_{SU(3)} \otimes \tau^c))$
  $\sim Tr(\tau^c) = 0$, using $\{\lambda^a, \lambda^b\} = \frac{1}{3}\delta^{ab} + d^{abc}\lambda^c$
- $U(1)\times SU(2)\times SU(3)$: $Tr(\{T^a_{SU(3)}, T^b_{SU(2)}\} Y) \sim Tr(\tau^b) Tr(\lambda^a) Y = 0$
- $U(1)\times U(1)\times SU(2)$: $Tr(\{Y, Y\}\tau^c) \sim Tr(\tau^c) = 0$ $\quad$ $Tr(\{Y,Y\}\tau^c)$
- $U(1)\times U(1)\times SU(3)$: $Tr(\{Y, Y\}\lambda^c) \sim Tr(\lambda^c) = 0$
- $SU(3)\times SU(3)\times SU(3)$:

Now: - $U(1)\times U(1)\times U(1)$: $Tr(YYY) = Y(f_i)^3 \rightsquigarrow \sum_{f_i} Y(f_i)^3 \stackrel{!}{=} 0$ as all fermions couple to $U(1)$

Include the sum by hand or already in it? How to get $Y(f_i)$ then? Only same f_i in loop

- $U(1)\times SU(3)\times SU(3)$: $Tr(\{\lambda^a, \lambda^b\} Y) \sim Tr((\frac{1}{3}\delta^{ab} + d^{abc}\lambda^c) Y) \sim Tr(Y)$
  $= Y(q_i) \rightsquigarrow \sum_{\text{quarks } q_i} Y(q_i) \stackrel{!}{=} 0$
  as only quarks couple
- $U(1)\times SU(2)\times SU(2)$: $Tr(\{\tau^a, \tau^b\} Y) \sim Tr(2\delta^{ab}\mathbb{1} Y) \sim Tr(Y)$
  $= Y(f_i)$ but as only left-handed fermions couple to $SU(2)$, we need to include $I_3(f_i)$, turning the singlets to zero.
  We need $I_3^2(f_i)$ because otherwise, the condition would be trivial.
  $\rightsquigarrow \sum_{f_i} I_3(f_i)^2 Y(f_i)$

b) The gauge group of Gravitation is $GL(4,\mathbb{R})$, which is a non-abelian gauge group. One then has to require that $GL(4) \times GL(4) \times$ ___ vanishes, while $GL(4) \times$ ___ $\times$ ___ is zero by triviality: $\sim \mathrm{Tr}(\{T^a, T^b\} G^c) \sim \mathrm{Tr}($

Why mixed gauge/grav. anomalies?

The non-vanishing contribution is $\sim \mathrm{Tr}(\{G^a, G^b\} Y) \sim \mathrm{Tr}(Y)$ and as the graviton couples to all fermions, we require $\sum_{f_i} Y(f_i) = 0$.

Not traceless G's?

Eq. (2) and (3) give 4 constraints on the Hypercharge. In the SM, we have 5 unknown Hypercharges for a family, i.e. for the q-dublett, the l-dublett, the u_R^c, d_R^c and e_R^c. As we do not have right-handed neutrino singlets, (only) the antiquark singlets can have an interchangeable Hypercharge and thus neither determine the Hypercharges, nor their ratios uniquely.

Neutrinos massless, don't couple to gravity?

$$\rightsquigarrow \quad 3\{Y_q + Y_q + Y_{\bar q_u} + Y_{\bar q_d}\} = 0$$

$$3\{Y_q + Y_q + Y_{\bar q_u} + Y_{\bar q_d}\} + Y_\ell + Y_\ell + Y_{\bar e} = 0$$

$$3\{\tfrac14 Y_q + \tfrac14 Y_q + \tfrac14 Y_\ell + \tfrac14 Y_\ell\} = 0$$

$$3\{Y_q^3 + Y_q^3 + Y_{\bar q_u}^3 + Y_{\bar q_d}^3\} + Y_\ell^3 + Y_\ell^3 + Y_{\bar e}^3 = 0$$

c) For $U_X(1)$, we get the same conditions as for $U(1)_Y$ and additionally from $U(1)_X \times U(1)_Y \times U(1)_Y$ and $U(1)_X \times U(1)_X \times U(1)_Y$ that

$$\sum_{f_i} Y(f_i)\, X(f_i)^2 = 0$$

$$\sum_{f_i} Y(f_i)^2\, X(f_i) = 0$$

This gives 10 constraints, but also 10 Hypercharges, making the solution unique. But as all constraints completely symmetric under $X \leftrightarrow Y$, this yields $X = Y$?