Disclaimer

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Advanced theoretical Particle Physics Homework 5

18.05.2018

1) Assume that the v.e.v. has the form

Transform like a 24 or given by a 24?

$$\langle \Sigma \rangle = \frac{v_x'}{2\sqrt{10}} \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ 0 & & & & -4 \end{pmatrix}, \text{ instead of}$$

$$\langle \Sigma \rangle = \frac{v_x}{2\sqrt{15}} \begin{pmatrix} 2 & & & & 0 \\ & 2 & & & \\ & & 2 & & \\ & & & -3 & \\ 0 & & & & -3 \end{pmatrix} \text{ like in class}$$

v_x' is a vector (→ matrix?) has a direction in field space? Or the matrix as v_x just a number?

We recall the generators t^a from class, which fulfill $tr(t^a t^b) = \frac{1}{2}\delta^{ab}$, $a,b \in \{1,\dots,24\}$

$$t^a = \begin{pmatrix} \lambda^a/2 & 0 \\ 0 & 0 \end{pmatrix} \text{ w/ } \lambda^a\text{: Gellmann-matrices for } a=1,\dots,8$$

$$t^{8+i} = \begin{pmatrix} 0 & 0 \\ 0 & \sigma^i/2 \end{pmatrix} \text{ w/ } \sigma^i\text{: Pauli-matrices for } i=1,2,3$$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$t^{12} = \frac{1}{2\sqrt{15}} \begin{pmatrix} 2 & & & & 0 \\ & 2 & & & \\ & & 2 & & \\ & & & -3 & \\ 0 & & & & -3 \end{pmatrix}$$

$$t^b = \frac{1}{2} \begin{pmatrix} 0 & A^b \\ A^{b\dagger} & 0 \end{pmatrix} \quad \text{for } b = 13,\dots,24$$

In t^{20} a $(-i)$? And $A^{b\dagger}$ or A^{bT}?

where
$$A^{13} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad A^{14} = \begin{pmatrix} i & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad A^{15} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A^{16} = \begin{pmatrix} 0 & 0 \\ i & 0 \\ 0 & 0 \end{pmatrix}$$

$$A^{17} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad A^{18} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ i & 0 \end{pmatrix}, \quad A^{19} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad A^{20} = \begin{pmatrix} 0 & -i \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$A^{21} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A^{22} = \begin{pmatrix} 0 & 0 \\ 0 & -i \\ 0 & 0 \end{pmatrix}, \quad A^{23} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad A^{24} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & -i \end{pmatrix}$$

No mass-term → unbroken? Why yes, if they get no mass, it's referred to as the symmetry being unbroken (remember v.e.v. acquiring ⇒ breaking gives mass)

We the check, which "particles" don't obtain a mass term, which means calculating $[\langle \Sigma \rangle, t^a]$ for the different generators

$a=1,\dots,8: \quad [\langle\Sigma\rangle, t^a] \propto \left[\begin{pmatrix}1&&&&0\\&1&&&\\&&1&&\\&&&1&\\0&&&&-4\end{pmatrix}, \begin{pmatrix}\lambda^a/2 & 0\\ 0 & 0\end{pmatrix}\right] = 0 \quad \forall a$

$i=1,2,3: \quad [\langle\Sigma\rangle, t^{8+i}] \propto \left[\begin{pmatrix}1&&&&\\&1&&&\\&&1&&\\&&&1&\\&&&&-4\end{pmatrix}, \begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&&\\0&0&0&\multicolumn{2}{c}{\sigma^i}\end{pmatrix}\right]$

$$= \begin{pmatrix}0 & 0\\ 0 & \left[\begin{pmatrix}1&0\\0&-4\end{pmatrix}, \sigma^i\right]\end{pmatrix} = \begin{pmatrix}\underset{3\times3}{0} & \underset{3\times2}{0}\\ \underset{2\times3}{0} & \underset{2\times2}{\begin{matrix}\sigma^i_{11} & \sigma^i_{12}\\ -4\sigma^i_{21} & -4\sigma^i_{22}\end{matrix}}\end{pmatrix} - \begin{pmatrix}0&0\\0&\begin{matrix}\sigma^i_{11} & -4\sigma^i_{12}\\ \sigma^i_{21} & -4\sigma^i_{22}\end{matrix}\end{pmatrix} = \begin{pmatrix}0&0\\0&\begin{matrix}0 & 5\sigma^i_{12}\\ -5\sigma^i_{21} & 0\end{matrix}\end{pmatrix}$$

vanishes only for σ_3

Easier way to see these commutators? Especially for the A's, as not possible as in lecture?

$a=12: \quad [\langle\Sigma\rangle, t^{12}] \propto \left[\begin{pmatrix}1&&&&0\\&1&&&\\&&1&&\\&&&1&\\0&&&&-4\end{pmatrix}, \begin{pmatrix}2&&&&0\\&2&&&\\&&2&&\\&&&-3&\\0&&&&-3\end{pmatrix}\right] = 0$

$b=13,\dots,24: \quad [\langle\Sigma\rangle, t^b] \propto \left[\begin{pmatrix}1&&&&\\&1&&&\\&&1&&\\&&&1&\\&&&&-4\end{pmatrix}, \begin{pmatrix}0 & A^k\\ A^{k\dagger} & 0\end{pmatrix}\right]$

$$= \begin{pmatrix}0&0&0&A^b_{11}&A^b_{12}\\0&0&0&A^b_{21}&A^b_{22}\\0&0&0&A^b_{31}&A^b_{32}\\A^{b\dagger}_{11}&A^{b\dagger}_{12}&A^{b\dagger}_{13}&0&0\\-4A^{b\dagger}_{21}&A^{b\dagger}_{22}&A^{b\dagger}_{23}&0&0\end{pmatrix} - \begin{pmatrix}0&0&0&A^k_{11}&-4A^k_{12}\\0&0&0&A^k_{21}&-4A^k_{22}\\0&0&0&A^k_{31}&-4A^k_{32}\\A^{k\dagger}_{11}&A^{k\dagger}_{12}&A^{k\dagger}_{13}&0&0\\A^{k\dagger}_{21}&A^{k\dagger}_{22}&A^{k\dagger}_{23}&0&0\end{pmatrix} = \begin{pmatrix}0&0&0&0&5A^k_{12}\\0&0&0&0&5A^k_{22}\\0&0&0&0&5A^k_{32}\\0&0&0&0&0\\-5A^{k\dagger}_{21}&-5A^{k\dagger}_{22}&-5A^{k\dagger}_{23}&0&0\end{pmatrix}$$

Need $A^k_{12} = A^k_{22} = A^k_{32} = 0 \rightsquigarrow b=13,\dots,18$ for commutator to vanish.

If $b=19,\dots,24 \rightsquigarrow$ doesn't vanish

$\Rightarrow 8+1+1+6 = 16$ generators

$\hat{=} (4^2-1)+1$ for $SU(4)\times U(1)$ unbroken.

Can also be seen from $\langle\Sigma\rangle \propto \begin{pmatrix}1&&&&0\\&1&&&\\&&1&&\\&&&1&\\0&&&&-4\end{pmatrix}$ $SU(4)$ $U(1)$

Can it really?

b) We have $V(\Sigma) = \mu_x^2 \, tr(\Sigma^2) + \frac{a}{4} \, tr(\Sigma^4) + \frac{b}{4} [tr(\Sigma^2)]^2$
which gives a mexican hat potential for $\mu_x^2 < 0$, as always.

Not really mexican hat, as now matrices? And not already conditions for a and b by this?

We calculate:

$$tr(\langle\Sigma\rangle^2) = \frac{v_x'^2}{40} \, tr\{diag(1,1,1,1,16)\} = \frac{v_x'^2}{2}$$

$$tr(\langle\Sigma\rangle^4) = \frac{v_x'^4}{1600} \, tr\{diag(1,1,1,1,256)\} = \frac{13}{80} v_x'^4$$

$$\rightsquigarrow V(\langle\Sigma\rangle) = \mu_x^2 \frac{v_x'^2}{2} + \frac{a}{4} \cdot \frac{13 v_x'^4}{80} + \frac{b}{4} \frac{v_x'^4}{4}$$

$$= \frac{1}{2} \mu_x^2 v_x'^2 + \frac{13a}{320} v_x'^4 + \frac{b}{16} v_x'^4$$

Not equal to $V(\langle\Sigma\rangle)$, different fct, but this is what is meant?

$$\rightsquigarrow V'(v_x') = \mu_x^2 v_x' + \frac{13a}{80} v_x'^3 + \frac{b}{4} v_x'^3 \overset{!}{=} 0$$

$$\overset{v_x' \neq 0}{\Longleftrightarrow} \underbrace{\mu_x^2}_{} = -\underset{>0}{v_x'^2} \underbrace{\left\{\frac{13a}{80} + \frac{b}{4}\right\}}_{>0}$$

$$\rightsquigarrow \frac{13a}{20} + b > 0$$

Constraints on the parameters? $\mu^2 > 0$ also possible as $\not\leq$?

c) Equivalent to $v_x'^2 = \dfrac{-\mu_x^2}{\frac{13a}{80} + \frac{b}{4}}$ for (2)'s minimum

For (1), we had in the lecture $v_x^2 = \dfrac{-\mu_x^2}{\frac{7a}{120} + \frac{b}{4}}$ for the minimum.

Why is (1) desired and (2) undesired?

We want to check under what conditions (1) is a deeper minimum than (2), i.e. $V(v_x') - V(v_x) \overset{!}{>} 0$

$$\rightsquigarrow v_x'^2 = \frac{-80\mu_x^2}{13a + 20b}, \quad v_x^2 = \frac{-120\mu_x^2}{7a + 30b}$$

$$\rightsquigarrow \frac{1}{2}\mu_x^2 v_x'^2 + \frac{13a}{320} v_x'^4 + \frac{b}{16} v_x'^4 - \frac{1}{2}\mu_x^2 v_x^2 - \frac{7a}{480} v_x^4 - \frac{b}{16} v_x^4 \overset{!}{>} 0$$

for $tr(\langle\Sigma\rangle^2) = \frac{1}{2} v_x^2$, $tr(\langle\Sigma\rangle^4) = \frac{7}{120} v_x^4$

$$\rightsquigarrow V(v_x) = \frac{1}{2}\mu_x^2 v_x^2 + \frac{7a}{480} v_x^4 + \frac{b}{16} v_x^4$$

this gives:

$$\frac{1}{2}\mu x^2 \left(\frac{-80\mu x^2}{13a+20b}\right) + \frac{13a}{320}\left(\frac{6400\mu x^4}{(13a+20b)^2}\right) + \frac{b}{16}\left(\frac{6400\mu x^4}{(13a+20b)^2}\right)$$

$$-\frac{1}{2}\mu x^2\left(\frac{-120\mu x^2}{7a+30b}\right) - \frac{7a}{480}\left(\frac{14400\mu x^4}{(7a+30b)^2}\right) - \frac{b}{16}\left(\frac{14400\mu x^4}{(7a+30b)^2}\right)$$

$$= \frac{1}{2}\mu x^4 \left\{\frac{120}{7a+30b} - \frac{80}{13a+20b}\right\} + \frac{1}{320}\,\frac{6400\mu x^4}{13a+20b} - \frac{1}{480}\,\frac{14400\mu x^4}{7a+30b}$$

$$= \frac{1}{2}\mu x^4 \left\{\frac{1560a+2400b-560a-2400b}{(7a+30b)(13a+20b)}\right\} + \frac{20\mu x^4}{13a+20b} - \frac{30\mu x^4}{7a+30b}$$

$$= \frac{\mu x^4}{2}\,\frac{500a}{(7a+30b)(13a+20b)} + \mu x^4\left\{\frac{140a+600b-390a-600b}{(13a+20b)(7a+30b)}\right\}$$

$$= \mu x^4 \left\{\frac{250a+250a}{(13a+20b)(7a+30b)}\right\} = 500a\,\mu x^4\,\frac{1}{\underbrace{(13a+20b)}_{>0 \text{ from b)}}\underbrace{(7a+30b)}_{>0 \text{ from lecture}}} \overset{!}{>} 0$$

Can use both constraints for a,b for both v.e.v

$\Rightarrow a > 0$

2)

a) For the gauge coupling running, we had derived

$$\frac{1}{g_i^2(Q)} = \frac{1}{g_i^2(M_Z)} + \frac{b_i}{8\pi^2}\ln\frac{Q}{M_Z}$$

Where from did eq. for those df_t etc. come?

$$\rightsquigarrow g_i^2(Q) = \frac{1}{\frac{1}{g_i^2(M_Z)} + \frac{b_i}{8\pi^2}\ln\frac{Q}{M_Z}}$$

Only for t or all u-like?

In the lecture, we had

$$\frac{df_t}{d\ln Q} = \frac{f_t}{16\pi^2}\left\{-3\left(\frac{8}{3}g_3^2 + \frac{3}{4}g_2^2 + \frac{17}{36}g_Y^2\right) + \frac{1}{2}\left(9f_t^2 + 3f_b^2 + 2f_\tau^2\right)\right\}$$

Integrate from M_X to Q, but gauge cpl. shown w/ M_Z?

$$\frac{df_b}{d\ln Q} = \frac{f_b}{16\pi^2}\left\{-3\left(\frac{8}{3}g_3^2 + \frac{3}{4}g_2^2 + \frac{5}{36}g_Y^2\right) + \frac{1}{2}\left(3f_t^2 + 9f_b^2 + 2f_\tau^2\right)\right\}$$

$$\frac{df_\tau}{d\ln Q} = \frac{f_\tau}{16\pi^2}\left\{-3\left(\frac{3}{4}g_2^2 + \frac{5}{4}g_Y^2\right) + \frac{1}{2}\left(6f_t^2 + 6f_b^2 + 5f_\tau^2\right)\right\}$$

where for now, we will neglect the $\sim f_t^2, f_b^2, f_\tau^2$ terms, as they will all scale with $\sim (cpl)^3$

Not only neglect 3 power of cpl. but also mix $f_t f_b^2$?

For f_t, we will have to solve,

$$\int_{M_X}^{Q}\frac{df_t}{f_t(\tilde{Q})} = \frac{1}{16\pi^2}\int_{\ln(M_X)}^{\ln(Q)} d\ln\tilde{Q}\left\{-3\left(\frac{8}{3}g_3^2 + \frac{3}{4}g_2^2 + \frac{17}{36}g_Y^2\right)\right\}$$

So we need to solve integrals of the form

Why handwriting now $\ln(M_X)$ to $\ln(Q)$ on r.h.s.?

$$\int_{\ln(M_X)}^{\ln(Q)} d\ln\tilde{Q}\, g_i^2(\tilde{Q}) = \int_{M_X}^{Q} d\tilde{Q}\, \frac{1}{\frac{\tilde{Q}}{g_i^2(M_Z)} + \frac{b_i}{8\pi^2}\tilde{Q}\ln\frac{\tilde{Q}}{M_Z}}$$

$$\frac{d\ln\tilde{Q}}{d\tilde{Q}} = \frac{1}{\tilde{Q}}$$

i.e. $\displaystyle\int dx \frac{1}{\frac{x}{c_1} + c_2 x\ln\frac{x}{c_3}} \overset{\text{Math.}}{=} \frac{\ln\left(c_2\ln\left(\frac{x}{c_3}\right) + \frac{1}{c_1}\right)}{c_2}$

Why we sol. of old eq. now? Aren't

$$= \frac{8\pi^2}{b_i}\left\{\ln\Bigg(\overbrace{\frac{b_i}{8\pi^2}\ln\frac{Q}{M_Z} + \frac{1}{g_i^2(M_Z)}}^{1/g_i^2(Q)}\Bigg) - \ln\Bigg(\overbrace{\frac{b_i}{8\pi^2}\ln\frac{M_X}{M_Z} + \frac{1}{g_i^2(M_Z)}}^{1/g_i^2(M_X)}\Bigg)\right\}$$

$$= \frac{8\pi^2}{b_i}\ln\frac{g_i^2(M_X)}{g_i^2(Q)}$$

this yields

$$\ln \frac{f_t(Q)}{f_t(M_X)} = -\frac{1}{16\pi^2}\left\{ 8\,\frac{8\pi^2}{b_3} \ln \frac{g_3^2(M_X)}{g_3^2(Q)} + \frac{9}{4}\,\frac{8\pi^2}{b_2} \ln \frac{g_2^2(M_X)}{g_2^2(Q)} + \frac{17}{12}\,\frac{8\pi^2}{b_Y} \ln \frac{g_Y^2(M_X)}{g_Y^2(Q)} \right\}$$

$$= \left\{ \frac{4}{b_3} \ln \frac{g_3^2(Q)}{g_3^2(M_X)} + \frac{9}{8b_2} \ln \frac{g_2^2(Q)}{g_2^2(M_X)} + \frac{17}{24 b_Y} \ln \frac{g_Y^2(Q)}{g_Y^2(M_X)} \right\}$$

$$\rightsquigarrow f_t(Q) = f_t(M_X) \left\{ \frac{g_3^2(Q)}{g_3^2(M_X)} \right\}^{4/b_3} \left\{ \frac{g_2^2(Q)}{g_2^2(M_X)} \right\}^{9/8b_2} \left\{ \frac{g_Y^2(Q)}{g_Y^2(M_X)} \right\}^{17/24b_Y}$$

$\alpha \propto g^2$ enough?

Analogously;

$$\ln \frac{f_b(Q)}{f_b(M_X)} = -\frac{1}{16\pi^2}\left\{ 8\,\frac{8\pi^2}{b_3} \ln \frac{g_3^2(M_X)}{g_3^2(Q)} + \frac{9}{4}\,\frac{8\pi^2}{b_2} \ln \frac{g_2^2(M_X)}{g_2^2(Q)} + \frac{5}{12}\,\frac{8\pi^2}{b_Y} \ln \frac{g_Y^2(M_X)}{g_Y^2(Q)} \right\}$$

$$= \left\{ \frac{4}{b_3} \ln \frac{g_3^2(Q)}{g_3^2(M_X)} + \frac{9}{8b_2} \ln \frac{g_2^2(M_X)}{g_2^2(Q)} + \frac{5}{24 b_Y} \ln \frac{g_Y^2(Q)}{g_Y^2(M_X)} \right\}$$

$$\rightsquigarrow f_b(Q) = f_b(M_X) \left\{ \frac{g_3^2(Q)}{g_3^2(M_X)} \right\}^{4/b_3} \left\{ \frac{g_2^2(Q)}{g_2^2(M_X)} \right\}^{9/8b_2} \left\{ \frac{g_Y^2(Q)}{g_Y^2(M_X)} \right\}^{5/24b_Y}$$

and

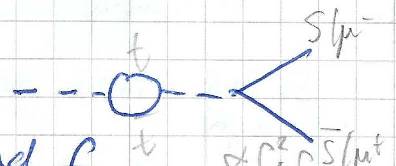
$$\ln \frac{f_\tau(Q)}{f_\tau(M_X)} = -\frac{1}{16\pi^2}\left\{ \frac{9}{4}\,\frac{8\pi^2}{b_2} \ln \frac{g_2^2(M_X)}{g_2^2(Q)} + \frac{15}{4}\,\frac{8\pi^2}{b_Y} \ln \frac{g_Y^2(M_X)}{g_Y^2(Q)} \right\}$$

$$= \left\{ \frac{9}{8b_2} \ln \frac{g_2^2(Q)}{g_2^2(M_X)} + \frac{15}{8b_Y} \ln \frac{g_Y^2(Q)}{g_Y^2(M_X)} \right\}$$

$$\rightsquigarrow f_\tau(Q) = f_\tau(M_X) \left\{ \frac{g_2^2(Q)}{g_2^2(M_X)} \right\}^{9/8b_2} \left\{ \frac{g_Y^2(Q)}{g_Y^2(M_X)} \right\}^{15/8b_Y}$$

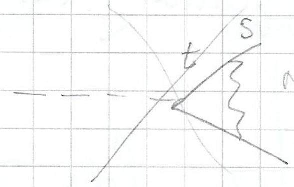
Why w/ b_Y now and not b_1?

b) The contributing Feynman-diagrams are



$\propto f_t^2 f_{s/\mu}$

and give an equal contribution to f_s and f_μ and thus cancel in f_s/f_μ.



No CKM mixing

CKM mixing → no transition between families? Only $t \leftrightarrow b$ etc.?

As $\ln f = [\int + \int + \int \ldots]$

$\rightsquigarrow f = \exp\{\ldots\}$ multiplicative

and cancels out.

c) $$\frac{df_{t,app}}{d\ln Q} = \frac{f_{t,app}(Q)}{16\pi^2}\left\{-3\left(\frac{8}{3}g_3^2 + \frac{3}{4}g_2^2 + \frac{17}{36}g_Y^2\right)\right\}$$

"app"? approximate

$$\frac{df_t}{d\ln Q} = \frac{f_t}{16\pi^2}\left\{-3\left(\frac{8}{3}g_3^2 + \frac{3}{4}g_2^2 + \frac{17}{36}g_Y^2\right) + \frac{9}{2}f_t^2\right\}$$

$$f_t^2(Q) = \frac{f_{t,app}^2(Q)}{F(Q)} \text{ ansatz} \rightsquigarrow f_t(Q) = \frac{f_{t,app}(Q)}{\sqrt{F(Q)}}$$

$$\rightsquigarrow \frac{df_t}{d\ln Q} = \frac{1}{\sqrt{F(Q)}}\frac{df_{t,app}(Q)}{d\ln Q} - \frac{1}{2}f_{t,app}\frac{1}{F(Q)^{3/2}}\frac{dF(Q)}{d\ln Q}$$

$$= \frac{f_{t,app}(Q)}{\sqrt{F(Q)}\,16\pi^2}\left\{-3\left(\frac{8}{3}g_3^2 + \frac{3}{4}g_2^2 + \frac{17}{36}g_Y^2\right)\right\} - \frac{1}{2}\frac{f_t(Q)}{F(Q)}\frac{dF(Q)}{d\ln Q}$$

Upper bound on the top quark mass?

$$= \frac{f_t(Q)}{16\pi^2}\left\{-3\left(\frac{8}{3}g_3^2 + \frac{3}{4}g_2^2 + \frac{17}{36}g_Y^2\right) - 8\pi^2\frac{1}{F(Q)}\frac{dF(Q)}{d\ln Q}\right\}$$

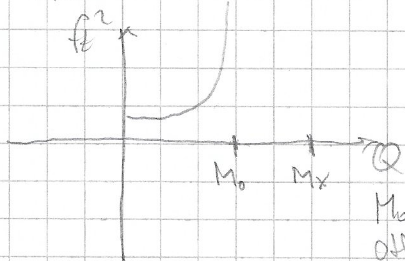
$$\rightsquigarrow -8\pi^2\frac{1}{F(Q)}\frac{dF(Q)}{d\ln Q} = \frac{9}{2}f_t^2(Q) = \frac{9}{2}\frac{f_{t,app}^2(Q)}{F(Q)}$$

$$\rightsquigarrow \frac{dF(Q)}{d\ln Q} = -\frac{9}{16\pi^2}f_{t,app}^2(Q)$$

$$\rightsquigarrow F = -\frac{9}{16\pi^2}\int d\ln Q\, f_{t,app}^2$$

$$\rightsquigarrow f_t^2(Q) = \frac{16\pi^2}{9}\frac{f_{t,app}^2(Q)}{\int_{\ln Q}^{\ln M_0} d\ln Q'\, f_{t,app}^2(Q')}$$

$f_t^2(M_X) < \infty$



M_0 has to be greater than M_X, otherwise pole in f_t^2.