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Theoretical Particle Physics 2 Homework 6 — Marvin Zanke

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1) $G_{LR} = SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_X$

Where did $U(1)_Y$ go? → $SU(2)_R \times U(1)_X$ breaks into $U(1)_Y$

$q_R = \begin{pmatrix} u_R \\ d_R \end{pmatrix}$, $l_R = \begin{pmatrix} \nu_R \\ e_R \end{pmatrix}$

$I_{3R}(u_R) = 1/2$, $I_{3R}(d_R) = -1/2$ etc.

Need to choose X s.t. eq. is fulfilled? Could also be a different formula if we choose X differently? → can absorb factor c into X (see tut.)

a) $Q = \underbrace{I_{3L} + I_{3R}}_{} + X$

Same for L and R, as one vanishes because it's a singlet

X: same for L and R doublets $X(q_R) = X(q_L)$, $X(l_L) = X(l_R)$

→ $Q(u_R) = Q(u_L)$ etc. follows instantly.

	u_R	d_R	u_L	d_L	ν_R	e_R	ν_L	e_L
I_{3L}	0	0	1/2	-1/2	0	0	1/2	-1/2
I_{3R}	1/2	-1/2	0	0	1/2	-1/2	0	0
Q	2/3	-1/3	2/3	-1/3	0	-1	0	-1
X	1/6	1/6	1/6	1/6	-1/2	-1/2	-1/2	-1/2

Read off what X in terms of B, L is? → yes (could also make a general ansatz but okay here)

$X = \frac{1}{2}(B-L)$

As $Q = I_{3L} + Y$ → $Y = I_{3R} + X$

b) $\mathcal{L}_{Yuk} = \sum_{\psi = q,l} y_\psi \bar{\psi}_L \phi \cdot \psi_R$, $(\phi \cdot \psi_R)_i = \phi_{i1}\psi_{R2} - \phi_{i2}\psi_{R1}$

singlet in $SU(2)_R$

doublet in $SU(2)_L$

Why $\phi \cdot \psi_R$ $SU(2)_R$ singlet? → group theory: have $(\phi \cdot \psi_R)$ which is doublet × anti doublet > $1 \otimes$ Adj → singlet under $SU(2)_R$

→ $\bar{\psi}_L \phi \cdot \psi_R$ is a singlet in $SU(2)_R$ and $SU(2)_L$

$SU(3)_C$ singlet because summed over color?

- $SU(3)_C$ singlet by construction as well
- For $U(1)_X$, consider $M = e^{i\hat{X}}$ → $\psi_R \to e^{iX(\psi_R)}\psi_R$

$\psi_L \to e^{iX(\psi_L)}\psi_L$

$\bar{\psi}_L \to e^{-iX(\psi_L)}\bar{\psi}_L$

$\phi \to e^{iX(\phi)}\phi$

$\Rightarrow X(\bar{\psi}_L) = -X(\psi_L) = -X(\psi_R)$

$\Rightarrow \mathcal{L}_{Yuk} \to \sum_{\psi=q,l} y_\psi \bar{\psi}_L \phi \cdot \psi_R \, e^{i(X(\psi_R) - X(\psi_L) + X(\phi))}$

and we need $X(\phi)=0$ for gauge invariance

c) $\sum_{f=q,l} y_f \overline{f_{Li}}\, \phi_{ij}\, f_{Rj} = \sum_{f=q,l} y_f \overline{f_{Li}} (\phi_{i1} f_{R2} - \phi_{i2} f_{R1})$

$$= \sum_{f=q,l} y_f \{ \overline{f_{L1}}\, \phi_{11} f_{R2} - \overline{f_{L1}}\, \phi_{12} f_{R1} + \overline{f_{L2}}\, \phi_{21} f_{R2} - \overline{f_{L2}}\, \phi_{22} f_{R1} \}$$

As $Q = I_{3L} + I_{3R} + X \overset{-X(\overline{f_L}) = X(f_R)}{\leadsto} Q(\overline{f_{Li}}) = -Q(f_{Ri})$

Not really true that $Q(\overline{f_{Li}}) = -Q(f_{Ri})$ e.g. $I_{3L}[\overline{U_L}] = -I_{3R}$

$\leadsto$ ϕ_{12} and ϕ_{21} electrically neutral

Also
$I_{3L}(\phi_{11}) = 1/2$ $\quad$ $I_{3R}(\phi_{11}) = 1/2$
$I_{3L}(\phi_{12}) = 1/2$ $\quad$ $I_{3R}(\phi_{12}) = -1/2$
$I_{3L}(\phi_{21}) = -1/2$ $\quad$ $I_{3R}(\phi_{21}) = 1/2$
$I_{3L}(\phi_{22}) = -1/2$ $\quad$ $I_{3R}(\phi_{22}) = -1/2$

$\leadsto$ $Q(\phi_{11}) = 1$, $Q(\phi_{12}) = 0$, $Q(\phi_{21}) = 0$, $Q(\phi_{22}) = -1$

It then follows $y_f \overline{f_{Li}}\, \phi_{ij}\, f_{Ri} \xrightarrow{i \neq j} y_f \overline{f_{Li}} \langle \phi_{ij} \rangle f_{Ri}$ still neutral

no summation

$y_f \overline{f_{Lj}}\, \phi_{jj}\, f_{Ri} \xrightarrow{i \neq j} y_f \overline{f_{Lj}} \langle \phi_{jj} \rangle f_{Ri}$ not neutral

For v.e.v. thus $\phi_{11} = 0 = \phi_{22}$, $\phi_{12} = v_1$, $\phi_{21} = v_2$

Doesn't break G_{LR} to $G_{SM} = SU(3)_c \times SU(2)_L \times U(1)_Y$

or $\quad SU(3)_c \times U(1)_{em}$

because $X(\phi)=0$ doesn't couple to gauge bosons. $y_f \overline{f_{Li}}\, \phi_{ij} f_{Ri} \to y_f \overline{f_{Li}} U_L^\dagger (U_L \phi_{ij} \times U_R^\dagger) U_R f_{Ri}$
$= y_f \overline{f_{Li}}\, \phi_{ij}\, f_{Ri}$

$SU(3)_c \times U(1)_{em}$ at even lower energies? But its el.m. unification at high scales?

Have to check $U_R \begin{pmatrix} 0 & v_1 \\ v_2 & 0 \end{pmatrix} U_R^\dagger = \begin{pmatrix} 0 & v_1 \\ v_2 & 0 \end{pmatrix}$?

d) $\Delta_L = (3,1)$ $SU(2)_L$ triplet, $SU(2)_R$ singlet

$\Delta_R = (1,3)$

$\tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Higgs fields; assume $X(\Delta_L) = X(\Delta_R) = 1$

$\tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$\tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$Q = I_{3L} + I_{3R} + X \rightsquigarrow Q(\Delta_L^1) = 2$

$|X\rangle = c_a |X_a\rangle \rightsquigarrow X_b|X\rangle = c_a X_b |X_a\rangle$

$X_b |X\rangle = c_a \langle X_c | X_b | X_a\rangle |X_c\rangle = c_a f_{bac} |X_c\rangle$ $Q(\Delta_L^2) = 1$

$= c_a | f_{bac} X_c\rangle = c_a | [X_b, X_a]\rangle \stackrel{!}{=} \lambda |X\rangle$

$\rightsquigarrow [X_b, X_a] = \lambda X_a \rightsquigarrow [X_b, X] = \lambda X$ $Q(\Delta_L^3) = 0$

Analogue for Δ_R

If instead basis s.t. $\Delta^+ = \frac{1}{2}(\Delta^1 + i\Delta^2)$

$\Delta^- = \frac{1}{2}(\Delta^1 - i\Delta^2)$

$\Delta^0 = \Delta^3$

$[\tau^3, \tau^\pm] = \pm \tau^\pm$

where $\tau^+ = \frac{1}{2}(\tau^1 + i\tau^2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$\tau^- = \frac{1}{2}(\tau^1 - i\tau^2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$\tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Use adjoint repr ≙ 3 dim. rep. of $SU(2)$ (see above) as then $\Delta_L = \Delta_L^+ \tau^+ + \Delta_L^- \tau^- + \Delta_L^0 \tau^0$

eigenvalue fixed for $\tau^\pm$ eigen"vector" of τ^3 $= \begin{pmatrix} \Delta_L^0 & \Delta_L^+ \\ \Delta_L^- & -\Delta_L^0 \end{pmatrix}$

$\Delta_L = \Delta_L^+ \tau^+ + \Delta_L^- \tau^- + \Delta_L^0 \tau^3$

~~$\rightsquigarrow \tilde{\tau}^+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tilde{\tau}^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tilde{\tau}^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ for $\begin{pmatrix} \Delta^+ \\ \Delta^0 \\ \Delta^- \end{pmatrix}$ triplet~~

$Q(\Delta) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ as $I_3(\Delta^+) = 1$, $I_3(\Delta^0) = 0$, $I_3(\Delta^-) = -1$

e) $\mathcal{L}_\Delta = g_L \overline{l_L^c}\, i\sigma_2 \Delta_L l_L + g_R \overline{l_R^c}\, i\sigma_2 \Delta_R l_R + \text{h.c.}$

"$SU(2)$ inv." tensor $i\sigma_2$?

- Don't have to check $SU(3)_c$, as we have no color carrying particles
- $SU(2)_L$ e.g.: $\Delta_L l_L \to U_L \Delta_L U_L^\dagger U_L l_L = U_L \Delta_L l_L$

doublet into antidoublet?

$\rightsquigarrow i\sigma_2 \Delta_L l_L \to U_L^* i\sigma_2 \Delta_L l_L$

and $\overline{l_L^c} \to \overline{l_L^c} U_L$

$\longrightarrow \overline{l_L^c}\, i\sigma_2 \Delta_L l_L \to \overline{l_L^c}\, i\sigma_2 \Delta_L l_L$ gauge invariant

Same U_L for Δ_L and l_L? $U_L - U_L^\dagger$ for Δ_L?

- $U(1)_X$: $\overline{l_L^c}\, i\sigma_2 \Delta_L l_L \to \overline{l_L^c}\, i\sigma_2 \Delta_L l_L e^{i(2X(l) + X(\Delta))}$

$= \overline{l_L^c}\, i\sigma_2 \Delta_L l_L$

as $2X(l) + X(\Delta_L) = 0$ Analogous for $SU(2)_R$

To find out which fermions can acquire a mass, we notice that $Q(\Delta_L^-) = I_{3L}(\Delta_L^-) + \cancel{I_{3R}(\Delta_L^-)} + X(\Delta^-) = 0$

$$Q(\Delta_L^0),\ Q(\Delta_L^+) \neq 0$$

s.t. only $\langle \Delta^-_{L,(R)} \rangle$ leaves $U(1)_{em}$ unbroken

But $Q(\Delta^-)$ should be (-1) and not 0?

$$\langle \Delta_L \rangle l_L = \begin{pmatrix} 0 & 0 \\ v_2 & 0 \end{pmatrix}\begin{pmatrix} \nu_L \\ e_L \end{pmatrix} = \begin{pmatrix} 0 \\ v_2 \nu_L \end{pmatrix}$$

$$i\sigma_2 \langle \Delta_L \rangle l_L = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ v_2 \nu_L \end{pmatrix} = \begin{pmatrix} v_2 \nu_L \\ 0 \end{pmatrix}$$

$$\rightarrow \overline{l_L^c}\, i\sigma_2 \Delta_L l_L = v_2 \overline{\nu_L^c} \nu_L$$

f) $|\langle \Delta_R \rangle| \gg |\langle \phi \rangle| \gg |\langle \Delta_L \rangle|$

$\uparrow$ breaks G_{LR} to G_{SM}

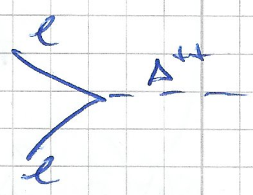
Why $SU(2)_R$ gauge bosons mass ~ scale of breaking?

$$Y(\langle \Delta_R \rangle) = I_3(\langle \Delta_R \rangle) + X(\langle \Delta_R \rangle) = 0$$

invariant under Y

Then G_{SM} breaks with ϕ to $SU(3)_c \times U(1)_{em}$

g) $$\Delta_L l_L = \begin{pmatrix} \Delta^+ & \Delta^{++} \\ \Delta^0 & -\Delta^+ \end{pmatrix}\begin{pmatrix} \nu_L \\ e_L \end{pmatrix} = \begin{pmatrix} \Delta^+ \nu_L + \Delta^{++} e_L \\ \Delta^0 \nu_L - \Delta^+ e_L \end{pmatrix}$$



$$\rightarrow i\sigma_2 \Delta_L l_L = \begin{pmatrix} \Delta^+ e_L - \Delta^0 \nu_L \\ \Delta^+ \nu_L + \Delta^{++} e_L \end{pmatrix}$$

$$\rightarrow \overline{l_L^c}\, \Delta_L^{++} l_L \subset \overline{l_L^c}\, i\sigma_2 \Delta_L l_L$$

2) (a) For quarks: $\mathcal{L}_{Yuk} = \sum_{i,j} \left(f^{(d)}_{jk} \bar{d}_{jR} \phi^\dagger q_{kL} - f^{(u)}_{jk} \bar{u}_{jR} \phi \cdot q_{kL} + h.c. \right)$

Including $U(1)_F$, this becomes

$$\tilde{\mathcal{L}}_{Yuk} = \sum_{i,j} \Big(f^{(d)}_{jk} \bar{d}_{jR} \phi^\dagger q_{kL} \left(\frac{f}{M}\right)^{F(q_k) - F(\phi) - F(d_j)}$$
$$- f^{(u)}_{jk} \bar{u}_{jR} \phi \cdot q_{kL} \left(\frac{f}{M}\right)^{F(q_k) + F(\phi) - F(u_j)} \Big) + h.c.$$

It's $+F(\phi)$, not $-F(\phi)$?

$$\xrightarrow[\left(\frac{f}{M}\right) = \epsilon]{\phi \to \binom{0}{v}} \sum_{i,j} \Big(f^{(d)}_{jk} \bar{d}_{jR}\, v\, d_{kL}\, \epsilon^{F(q_k) - F(\phi) - F(d_j)}$$
$$- f^{(u)}_{jk} \bar{u}_{jR}\, v\, u_{kL}\, \epsilon^{F(q_k) + F(\phi) - F(u_j)}$$

$\phi \to \binom{0}{v}$ or $\phi \to v$?

$$F(\phi) = 0, \quad F(q_3) = 0, \quad F(q_2) = -1, \quad F(q_1) = -2$$

$$\longrightarrow \mathcal{M}^{(d)} = v \begin{pmatrix} f^{(d)}_{11} & f^{(d)}_{12}\epsilon^{-1} & f^{(d)}_{13}\epsilon^{-2} \\ f^{(d)}_{21}\epsilon & f^{(d)}_{22} & f^{(d)}_{23}\epsilon^{-1} \\ f^{(d)}_{31}\epsilon^{2} & f^{(d)}_{32}\epsilon & f^{(d)}_{33} \end{pmatrix}$$

$$\mathcal{M}^{(u)} = v \begin{pmatrix} f^{(u)}_{11} & f^{(u)}_{12}\epsilon^{-1} & f^{(u)}_{13}\epsilon^{-2} \\ f^{(u)}_{21}\epsilon & f^{(u)}_{22} & f^{(u)}_{23}\epsilon^{-1} \\ f^{(u)}_{31}\epsilon^{2} & f^{(u)}_{32}\epsilon & f^{(u)}_{33} \end{pmatrix}$$

No second term as for quarks?

For leptons: $\mathcal{L}_{Yuk} = f^{(l)}_j \bar{l}_{jR} \phi^\dagger L_{jL}$

already diagonal $\rightsquigarrow$ $\mathcal{M}^{(l)} = v \begin{pmatrix} f^{(l)}_1 & 0 & 0 \\ 0 & f^{(l)}_2 & 0 \\ 0 & 0 & f^{(l)}_3 \end{pmatrix}$

Same SM charges as l.h./r.h. SM fermions?

b) Introduce F_L, F_L^c, F_R, F_R^c same SM charges as l.h./r.h. SM fermions f_{Li}, f_{Ri}, $i = 1,2$

Possible, gauge invariant mass terms: $M \bar{F}_L F_R^c$ and $M \bar{F}_R F_L^c$ if $U(1)_F$ charge s.t. they are invariant, i.e.

$F(F_L) + F(F_R) = 0$ ← flavon field

why f heavier SM generation?

Also: $h \bar{f}_{R,2} f_{L,2}$, $f \bar{f}_{R1} F_R$, $f \bar{f}_{L1} F_L$, $h \bar{f}_{R1} F_L$, $h \bar{F}_R f_{L1}$ allowed, while other SM Yukawa couplings forbidden.

why other SM cpl. forbidden?

$\vartheta = \langle h \rangle$

$\Longrightarrow$

$$(\overline{f}_{1L}, \overline{f}_{1R}, \overline{F}_L, \overline{F}_R, \overline{f}_{2L}, \overline{f}_{2R}) \begin{pmatrix} 0 & 0 & \langle f \rangle & \vartheta & 0 & 0 \\ 0 & 0 & \vartheta & \langle f \rangle & 0 & 0 \\ \langle f \rangle & \vartheta & 0 & M & 0 & 0 \\ \vartheta & \langle f \rangle & M & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \vartheta \\ 0 & 0 & 0 & 0 & \vartheta & 0 \end{pmatrix} \begin{pmatrix} f_{1L} \\ f_{1R} \\ F_L \\ F_R \\ f_{2L} \\ f_{2R} \end{pmatrix}$$

If we have $\overline{p}_{L1}$ $\dot{F}_L$, then also $F_L f_{L1}$?