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Advanced theoretical Particle Physics Homework 8

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1) a) We consider $[Q_a, P_\mu]$, $a \in \{1,2,3,4\}$

Q_a: Maj. spinor of ferm. supercharges

P_μ: generator of translations, i.e. the 4-mom., μ a Lorentz index

Maj. super-particle should be bosonic? $\leadsto$ It's only an operator $Q|f\rangle = |b\rangle$ etc. particles layer

From the lecture, we know that $[\text{Even}, \text{Odd}] = \text{Odd}$, where P_μ, $M_{\mu\nu}$ are even and Q is odd

Why Q odd, P_μ even etc. and com. must be odd? $\leadsto$ P_μ bosonic $\leftrightarrow$ even, need Q fermionic $\leftrightarrow$ odd and thus also partial com.

thus $[Q_a, P_\mu] \sim Q$. We also need one remaining spinor index a and Lorentz index μ.

Basis of Dirac space? $\leadsto$ Dirac algebra

The available γ-matrices (16, basis of the Dirac space) are

$$\mathbb{1},\ \gamma_5,\ \gamma_\mu,\ \gamma_\mu\gamma_5,\ \sigma_{\mu\nu} \equiv \frac{i}{2}[\gamma_\mu, \gamma_\nu]$$

and thus, only γ_μ and $\gamma_\mu\gamma_5$ are possible because of their desired Lorentz / Dirac structure.

Why e.g. P_μ itself forbidden – odd × even = odd? And C_{ab} would also be possible? And γ_μ even? Needs to be defined as $\gamma_\mu \in M_{\mu\nu}$? $\leadsto$ To be cleared up: odd × even = odd; γ_μ not well-defined as it's a repr. and even / odd only defined for generators

$$\rightarrow [Q_a, P_\mu] = (c_1\gamma_\mu + c_2\gamma_\mu\gamma_5)_{ab}\, Q_b$$

b) We will now use that Q is a Majorana spinor, i.e.

$$Q_a = C_{ab}\bar{Q}_b \qquad (Q^c = C\bar{Q}^T \stackrel{!}{=} Q)$$

$$\Leftrightarrow (C^{-1})_{ca} Q_a = \bar{Q}_c$$

From the lecture, we also know $C^{-1} = C^\dagger$, $C^T = -C$

and $C\,\Gamma^T C^{-1} = \eta_\Gamma \Gamma$, where $\eta_\Gamma = +1$ for $\Gamma \in \{\mathbb{1}, \gamma_5, \gamma_\mu\gamma_5\}$

$\eta_\Gamma = -1$ for $\Gamma \in \{\gamma_\mu, \sigma_{\mu\nu}\}$

with $C = i\gamma^2\gamma^0$.

It also quickly follows: $C^2 = -\mathbb{1}$, $\Gamma^T = \eta_\Gamma C^{-1}\Gamma C = \eta_\Gamma C \Gamma C^{-1}$

We start by taking the hermitian conjugate of the commutator:

Why P_μ herm.?

$$[Q_a, P_\mu]^\dagger = (c_1\gamma_\mu + c_2\gamma_\mu\gamma_5)_{ab}^{\ *}\, Q_b^\dagger$$

$$\Leftrightarrow [Q_a^\dagger, P_\mu] = (c_1^*\gamma_\mu^\dagger + c_2^*\gamma_5^\dagger\gamma_\mu^\dagger)_{ba}\, Q_b^\dagger$$

$$= (c_1^*\gamma_0\gamma_\mu\gamma_0 + c_2^*\gamma_5\gamma_0\gamma_\mu\gamma_0)_{ba}\, Q_b^\dagger$$

$| \cdot (\gamma_0)_{ac}$

$\{\gamma_\mu, \gamma_5\} = 0$

$$\Leftrightarrow [\bar{Q}_c, P_\mu] = (\gamma^0)_{bd} (C_1^* \gamma_\mu \gamma_0 \gamma_0 - C_2^* \gamma_5 \gamma_\mu \gamma_0 \gamma_0)_{dc} Q_b^\dagger$$

$$= \bar{Q}_d (C_1^* \gamma_\mu - C_2^* \gamma_5 \gamma_\mu)_{dc}$$

Maj. Cond.

$$\Rightarrow (C^{-1})_{ce} [Q_e, P_\mu] = (C^{-1})_{de}\, Q_e (C_1^* \gamma_\mu + C_2^* \gamma_\mu \gamma_5)_{dc}$$

$$\Leftrightarrow (C^{-1})_{ce} (C_1 \gamma_\mu + C_2 \gamma_\mu \gamma_5)_{eb} Q_b = (C^{-1})_{de} (C_1^* \gamma_\mu + C_2^* \gamma_\mu \gamma_5)_{dc} Q_e$$

$$= (C^{-1})_{de} (C_1^* \gamma_\mu^T + C_2^* \gamma_5^T \gamma_\mu^T)_{cd} Q_e$$

$$= (C^{-1})_{de} (-C_1^* C^{-1} \gamma_\mu C - C_2^* C^{-1} \gamma_5 \gamma_\mu C)_{cd} Q_e$$

$$= (C^{-1})_{de} (C)_{fd} (C^{-1})_{cg} (-C_1^* \gamma_\mu + C_2^* \gamma_\mu \gamma_5)_{gf} Q_e$$

$$= (C^{-1})_{cg} (-C_1^* \gamma_\mu + C_2^* \gamma_\mu \gamma_5)_{ge} Q_e$$

$$\Leftrightarrow (C^{-1})_{ce} (C_1 \gamma_\mu + C_2 \gamma_\mu \gamma_5)_{eb} Q_b = (C^{-1})_{ce} (-C_1^* \gamma_\mu + C_2^* \gamma_\mu \gamma_5)_{eb} Q_b$$

Do γ_0 and P_μ commute? $\rightsquigarrow$ yes, different space

$\rightsquigarrow$ $C_1^* = -C_1$, $C_2^* = C_2$

$\rightsquigarrow$ $C_1 = ix$ purely imaginary

$C_2 = y$ purely real

c)

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]]$$

$$= A[B,C] - [B,C]A + B[C,A] - [C,A]B + C[A,B] - [A,B]C$$

$$= ABC - ACB - BCA + CBA + BCA - BAC - CAB + ACB$$
$$+ CAB - CBA - ABC + BAC = 0$$

d)

Setting $A = P_\mu$, $B = P_\nu$, $C = Q_a$, we find

$$[P_\mu, [P_\nu, Q_a]] = -[P_\nu, [Q_a, P_\mu]] - [Q_a, \underbrace{[P_\mu, P_\nu]}_{=0}]$$

$$\rightsquigarrow [P_\mu, [P_\nu, Q_a]] = -[P_\nu, [Q_a, P_\mu]]$$

l.h.s: $[P_\mu, [P_\nu, Q_a]] = +[(C_1 \gamma_\nu + C_2 \gamma_\nu \gamma_5)_{ab} Q_b, P_\mu]$

$$= (C_1 \gamma_\nu + C_2 \gamma_\nu \gamma_5)_{ab} (C_1 \gamma_\mu + C_2 \gamma_\mu \gamma_5)_{bc} Q_c$$

$$= (C_1^2 \gamma_\nu \gamma_\mu + C_2^2 \gamma_\nu \overbrace{\gamma_5 \gamma_\mu}^{(-1)} \gamma_5 + C_1 C_2 \gamma_\nu \gamma_\mu \gamma_5 + C_1 C_2 \gamma_\nu \overbrace{\gamma_5 \gamma_\mu}^{(-1)})_{ac} Q_c$$

$$= (C_1^2 \gamma_\nu \gamma_\mu - C_2^2 \gamma_\nu \gamma_\mu)_{ac} Q_c$$

r.h.s. : $-[P_\nu, [Q_a, P_\mu]] = [(c_1\gamma_\mu + c_2\gamma_\mu\gamma_5)_{ab} Q_b, P_\nu]$

$$= (c_1\gamma_\mu + c_2\gamma_\mu\gamma_5)_{ab}(c_1\gamma_\nu + c_2\gamma_\nu\gamma_5)_{bc} Q_c$$

$$= (c_1^2\gamma_\mu\gamma_\nu + c_2^2\gamma_\mu\overbrace{\gamma_5\gamma_\nu}^{(-1)}\gamma_5 + c_1c_2\gamma_\mu\overbrace{\gamma_\nu\gamma_5}^{(-1)} + c_1c_2\gamma_\mu\gamma_5\gamma_\nu)_{ac} Q_c$$

$$= (c_1^2\gamma_\mu\gamma_\nu - c_2^2\gamma_\mu\gamma_\nu)_{ac} Q_c$$

this yields: $(c_1^2\gamma_\nu\gamma_\mu - c_2^2\gamma_\nu\gamma_\mu)_{ac} Q_c = (c_1^2\gamma_\mu\gamma_\nu - c_2^2\gamma_\mu\gamma_\nu)_{ac} Q_c$

$$\Leftrightarrow c_1^2[\gamma_\nu, \gamma_\mu]_{ac} Q_c = c_2^2[\gamma_\nu, \gamma_\mu]_{ac} Q_c$$

$\rightsquigarrow c_1^2 = c_2^2$ in general, as $[\gamma_\nu, \gamma_\mu] \propto \sigma_{\nu\mu} \neq 0$ and $Q \neq 0$

e) We thus have $c_1 = i\overset{\text{real}}{x}$, $c_2 = \overset{\text{real}}{y}$ and $c_1^2 = c_2^2$

$\rightsquigarrow -x^2 = y^2$, which is only possible for real x, y, if

$x = 0 = y \rightsquigarrow c_1 = 0$, $c_2 = 0$ $\longrightarrow [Q_a, P_\mu] = 0$

Why anticomm. for (odd, odd)? $\rightsquigarrow$ because they are fermionic
$O|b\rangle \propto |b\rangle$
$O|f\rangle \propto -|f\rangle$

f) Now consider $\{Q_a, Q_b\}$, $a, b \in \{1, ..., 4\}$
$\uparrow$ $\uparrow$ SUSY Generators

In the lecture, we had $\{\text{odd}, \text{odd}\} = \text{Even}$

Thus, $M_{\mu\nu}$ and P_μ are the only possibilities out of the Poincaré alg.
$\uparrow$ generate rotations and Lorentz-boosts

We also need something symmetric under $a \leftrightarrow b$, as $\{Q_a, Q_b\}$ is symmetric under $a \leftrightarrow b$ and we don't want any Lorentz-indices.
Thus, for P_μ, we need to contract w/ γ^μ and for $M_{\mu\nu}$ w/ $\sigma^{\mu\nu}$.
There are 2 free spinor indices.

why not w/o C or even only C_{ab}? Any other possibilities? $\rightsquigarrow$ not symmetric

We claim that $(\gamma_\mu C)_{ab}$, $(\sigma_{\mu\nu}C)_{ab}$ are symmetric under $a \leftrightarrow b$, while $(\gamma_\mu\gamma_5 C)_{ab}$ isn't.

- $(\gamma_\mu C)^T = C^T\gamma_\mu^T = -C(-C^{-1}\gamma_\mu C) = \gamma_\mu C$

$\bullet\ (\sigma_{\mu\nu} C)^T = \frac{i}{4} C^T [\gamma_\nu^T, \gamma_\mu^T] = -\frac{i}{4} C [-C^{-1}\gamma_\nu C, -C^{-1}\gamma_\mu C]$
$= -\frac{i}{4} C C^{-1} [\gamma_\nu, \gamma_\mu] C = \frac{i}{4} [\gamma_\mu, \gamma_\nu] C = \sigma_{\mu\nu} C$

$\bullet\ (\gamma_\mu \gamma_5 C)^T = C^T (\gamma_\mu\gamma_5)^T = -C\,(C^{-1}\gamma_\mu\gamma_5 C) = -\gamma_\mu\gamma_5 C$

$\rightsquigarrow \{Q_a, Q_b\} = C_3 (\gamma^\mu C)_{ab} P_\mu + C_4 (\sigma^{\mu\nu} C)_{ab} M_{\mu\nu}$
as the general form

Important in which rep.? $\rightsquigarrow$ yes, but it's okay to choose phys. useful relations!

g) $\{A, [B,C]\} + \{B, [A,C]\} + [C, \{A,B\}]$
$= A[B,C] + [B,C]A + B[A,C] + [A,C]B + C\{A,B\} - \{A,B\}C$
$= \underline{ABC} - ACB + BCA - CBA + BAC - BCA + ACB - CAB$
$+ CAB + CBA - \underline{ABC} - BAC$

h) $A = Q_a,\ B = Q_b,\ C = P_\mu$
$\rightsquigarrow [P_\mu, \{Q_a, Q_b\}] = -\{Q_a, \underbrace{[Q_b, P_\mu]}_{=0}\} - \{Q_b, \underbrace{[Q_a, P_\mu]}_{=0}\} = 0$

$\rightsquigarrow 0 = [P_\mu, C_3(\gamma^\nu C)_{ab} P_\nu + C_4 (\sigma^{\nu\lambda} C)_{ab} M_{\nu\lambda}]$
$= C_3 (\gamma^\nu C)_{ab} \underbrace{[P_\mu, P_\nu]}_{=0} + C_4 (\sigma^{\nu\lambda} C)_{ab} \underbrace{[P_\mu, M_{\nu\lambda}]}_{\neq 0 \text{ in general}}$

$\rightsquigarrow C_4 = 0$
$\rightsquigarrow \{Q_a, Q_b\} = C_3 (\gamma^\mu C)_{ab} P_\mu$

i) $Q' = \overset{\text{real}}{y} Q$ is another Majorana spinor that can be used to represent the SUSY generator

$(Q')^c = C \overline{Q'}^T = C (y\overline{Q})^T = y C \overline{Q}^T = y Q^c$

why another Maj. spinor? $\rightsquigarrow Q^c \overset{!}{=} Q$ fulfilled and can also multiply by phase

$\rightsquigarrow \{Q'_a, Q'_b\} = y^2 \{Q_a, Q_b\} = y^2 C_3 (\gamma^\mu C)_{ab} P_\mu$
$= C_3' (\gamma^\mu C)_{ab} P_\mu$

$\rightsquigarrow C_3' = y^2 C_3$ can be scaled as desired

Can still be complex? $\rightsquigarrow$ yes, but $C_3' = y^2 C_3$ and thus only rescaling

2) a) Consider supermultiplets as irreducible reps. of the SUSY algebra.

Have $Q|\text{fermion}\rangle \equiv Q|f\rangle = |\text{boson}\rangle \equiv |b\rangle$

and Q

$Q_a|\ldots\rangle$ + $Q_b|\ldots\rangle$? ⇝ yes.

From 1), we know $[Q_a, P_\mu] = 0 = Q_a P_\mu - P_\mu Q_a$

Q or Q_a? ⇝ with Q

$$\rightsquigarrow 0 = P^\mu Q_a P_\mu - P^\mu P_\mu Q_a = Q_a P^2 - P^2 Q_a = [Q_a, P^2]$$

Same?

where $P^2 = E^2 - \vec{p}^2 \hat{=} m^2$

$Q^2|f\rangle \neq |f\rangle$ change between fer.? ⇝ yes, can change through whole multiplet

then $Q P^2|f\rangle = P^2 Q|f\rangle$

$$\Leftrightarrow Q m_f^2|f\rangle = P^2|b\rangle = m_b^2|b\rangle = m_b^2 Q|f\rangle = Q m_b^2|f\rangle$$

$$\Leftrightarrow m_f = m_b$$

Can also be done w/ P_μ instead of P^2 by setting $\vec{p} = 0$? ⇝ done w/ P^2 in tutorial, used that P^2 is Casimir op. and commutes w/ everything

And also $P^2 Q^2|f\rangle = Q^2 P^2|f\rangle$

$$\Leftrightarrow P^2|f'\rangle = Q^2 m_f^2|f\rangle$$

$$\Leftrightarrow m_{f'}^2 Q^2|f\rangle = m_f^2 Q^2|f\rangle$$

and $P^2 Q^2|b\rangle = Q^2 P^2|b\rangle$

$$\Leftrightarrow P^2|b'\rangle = Q^2 m_b^2|b\rangle$$

$$\Leftrightarrow m_{b'}^2 Q^2|b\rangle = m_b^2 Q^2|b\rangle$$

b) Consider a supermultiplet w/ fixed 4-mom. P_μ leaves the number of fermionic and bosonic d.o.f. unchanged.

Same P_μ for each comp. of the supermultiplet? ⇝ in this proof yes, assumed this case

Also we will use from 1): $\{Q_a, Q_b\} = C_3(\gamma^\mu C)_{ab} P_\mu$

Have $\sum_{b=1}^{N_b} \langle b|b\rangle - \sum_{f=1}^{N_f} \langle f|f\rangle = N_b - N_f$

Consider $C_3(\gamma^\mu C)_{ab} P_\mu \{N_b - N_f\} = C_3(\gamma^\mu C)_{ab} P_\mu \{\sum_{b=1}^{N_b} \langle b|b\rangle - \sum_{f=1}^{N_f} \langle f|f\rangle\}$

Can pull P_μ out of braket? ⇝ yes, eigenvalue

$$= C_3(\gamma^\mu C)_{ab}\{\sum_{b=1}^{N_b} \langle b|P_\mu|b\rangle - \sum_{f=1}^{N_f} \langle f|P_\mu|f\rangle\}$$

$$= \sum_{b=1}^{N_b} \langle b|\{Q_a, Q_b\}|b\rangle - \sum_{f=1}^{N_f} \langle f|\{Q_a, Q_b\}|f\rangle$$

$$= \sum_{i=1}^{N} \sum_{b=1}^{N_b} \langle b|Q_a|i\rangle\langle i|Q_b|b\rangle + \langle b|Q_b|i\rangle\langle i|Q_a|b\rangle$$

$$- \sum_{i=1}^{N} \sum_{f=1}^{N_f} \langle f|Q_a|i\rangle\langle i|Q_b|f\rangle + \langle f|Q_b|i\rangle\langle i|Q_a|f\rangle$$

$$= \sum_{f=1}^{N_f}\sum_{b=1}^{N_b} \langle b|Q_a|f\rangle\langle f|Q_b|b\rangle + \langle b|Q_b|f\rangle\langle f|Q_a|b\rangle$$
$$- \sum_{b=1}^{N_b}\sum_{f=1}^{N_f} \langle f|Q_a|b\rangle\langle b|Q_b|f\rangle + \langle f|Q_b|b\rangle\langle b|Q_a|f\rangle$$

$$= 0$$

If $P_\mu \neq 0$, the first step (multiplying w/ P_μ is okay w/o loosing information $\Rightarrow N_b - N_f = 0 \Leftrightarrow N_b = N_f$

c) For the ground state, $P_\mu = 0$, then $N_b - N_f$ is some number, called the Witten index.

there could be an other proof?
→ yes, not complete in this sense

vanishing energy & momentum?