Hinweis

Die vorliegende Lösung wurde im Rahmen der jeweiligen Lehrveranstaltung an der Universität Bonn erstellt. Sofern im oberen Teil der ersten Seite oder auf der unten angegebenen Webseite nicht anders vermerkt, wurde diese Lösung von mir, Marvin Zanke, alleine angefertigt und eingereicht. Bei allem in einer anderen Farbe als dem üblichen Blau handelt es sich in der Regel um Korrekturen von mir oder des Tutors. Für mehr Informationen und meine gesamten Unterlagen, siehe:
https://www.physics-and-stuff.com/

Ich erhebe keinen Anspruch auf Richtigkeit und Vollständigkeit der vorliegenden Lösungen! Dies gilt ebenso für obengenannte Korrekturen.

Übung 7 bei Florian

Physik IV Blatt 3

Marvin Zanke

1	2	3	4	Σ
5	5	7.5	4	15.5/20

Kfz-Hilfen

1) $$-\frac{\hbar^2}{2m}\nabla^2\psi(\vec r)+V(\vec r)\psi(\vec r)=E\psi(\vec r)$$

$$\psi(\vec r)=\phi_x(x)\phi_y(y)\phi_z(z)$$

$$\Leftrightarrow -\frac{\hbar^2}{2m}\nabla^2(\phi_x(x)\phi_y(y)\phi_z(z))+V(\vec r)\phi_x(x)\phi_y(y)\phi_z(z)=E\phi_x(x)\phi_y(y)\phi_z(z)$$

$$\Leftrightarrow -\frac{\hbar^2}{2m}\left[\phi_y(y)\phi_z(z)\frac{\partial^2}{\partial x^2}\phi_x(x)+\phi_x(x)\phi_z(z)\frac{\partial^2}{\partial y^2}\phi_y(y)+\phi_x(x)\phi_y(y)\frac{\partial^2}{\partial z^2}\phi_z(z)\right]$$

$$+V(\vec r)\phi_x(x)\phi_y(y)\phi_z(z)=E\phi_x(x)\phi_y(y)\phi_z(z)$$

$$\Leftrightarrow -\frac{\hbar^2}{2m}\left[\frac{1}{\phi_x(x)}\frac{\partial^2}{\partial x^2}\phi_x(x)+\frac{1}{\phi_y(y)}\frac{\partial^2}{\partial y^2}\phi_y(y)+\frac{1}{\phi_z(z)}\frac{\partial^2}{\partial z^2}\phi_z(z)\right]$$

$$+V(\vec r)=E$$

3 Diffglg. ① $-\frac{\hbar^2}{2m}\left[\frac{1}{\phi_x(x)}\frac{\partial^2}{\partial x^2}\phi_x\right]+V(\vec r)=E$

② $-\frac{\hbar^2}{2m}\left[\frac{1}{\phi_y(y)}\frac{\partial^2}{\partial y^2}\phi_y\right]+V(\vec r)=E$

③ $-\frac{\hbar^2}{2m}\left[\frac{1}{\phi_z(z)}\frac{\partial^2}{\partial z^2}\phi_z\right]+V(\vec r)=E$

Wir suchen die Lösungen ~~für~~ jeder der Komponenten

$$-\frac{\hbar^2}{2m}\left[\frac{1}{\phi_{x_i}(x_i)}\frac{\partial^2}{\partial x_i^2}\phi_{x_i}\right]=E-V(\vec r)$$

Ansatz: $\phi_{x_i}=e^{\lambda x_i}$ $\Rightarrow \frac{\partial}{\partial x_i}\phi_{x_i}=\lambda e^{\lambda x_i}$, $\frac{\partial^2}{\partial x_i^2}\phi_{x_i}=\lambda^2 e^{\lambda x_i}$

$$\Rightarrow -\frac{\hbar^2}{2m}\left[e^{-\lambda x_i}\cdot\lambda^2 e^{\lambda x_i}\right]=E-V(\vec r)$$

$$\Leftrightarrow \lambda^2=\frac{V(\vec r)-E}{\hbar^2}\cdot 2m$$

Da uns nur Lösungen interessieren mit $V(\vec r)=0$ für $0<x,y,z<L$ und außen gilt $V(\vec r)=\infty$, folgt für λ:

$$\lambda=\pm i\frac{\sqrt{2mE}}{\hbar}$$

$$\Rightarrow \phi_{x_i}(x_i)=Ae^{i\sqrt{\frac{2mE}{\hbar^2}}x_i}+Be^{-i\sqrt{\frac{2mE}{\hbar^2}}x_i}$$

Es gilt: $\phi_{x_i}(0) = 0 = \phi_{x_i}(L)$ und damit:

$$0 = A \cdot e^{0} + B e^{-0} \Leftrightarrow A = -B \Rightarrow \phi_{x_i}(x_i) = A\left(e^{i\sqrt{\frac{2mE}{\hbar}}x_i} - e^{-i\sqrt{\frac{2mE}{\hbar}}x_i}\right)$$

$$0 = A\left(e^{i\sqrt{\frac{2mE}{\hbar^2}}L} - e^{-i\sqrt{\frac{2mE}{\hbar^2}}\cdot L}\right) \quad = 2iA \sin\left(\sqrt{\frac{2mE}{\hbar^2}}x_i\right)$$

$$= 2iA \sin\left(\sqrt{\frac{2mE}{\hbar^2}}L\right) \overset{(\checkmark)}{} \Rightarrow \sqrt{\frac{2mE}{\hbar^2}} \cdot L \overset{(*)}{=} n_{x_i}\pi$$

$$\Leftrightarrow 2mEL^2 = \hbar^2 \pi^2 n_{x_i}^2$$

$$\Leftrightarrow E = \frac{\hbar^2}{2m} \cdot n_{x_i}^2 \pi^2 \cdot \frac{1}{L^2} \checkmark$$

$$\Psi(r) = \phi_x(x)\phi_y(y)\phi_z(z) = 2iA \sin\left(\sqrt{\frac{2mE}{\hbar^2}}x\right) \cdot 2iA \sin\left(\sqrt{\frac{2mE}{\hbar^2}}y\right) \cdot 2iA \sin\left(\sqrt{\frac{2mE}{\hbar^2}}z\right)$$

$$= -8iA^3 \sin\left(\sqrt{\frac{2mE}{\hbar^2}}x\right) \sin\left(\sqrt{\frac{2mE}{\hbar^2}}y\right) \sin\left(\sqrt{\frac{2mE}{\hbar^2}}z\right)$$

Normierung: $1 = \int |\Psi(\vec r)|^2 dx\,dy\,dz = 64|A|^6 \int_0^L\int_0^L\int_0^L \sin^2\left(\sqrt{\frac{2mE}{\hbar^2}}x\right) \sin^2\left(\sqrt{\frac{2mE}{\hbar^2}}y\right) \sin^2\left(\sqrt{\frac{2mE}{\hbar^2}}z\right) dx\,dy\,dz$

Nun gilt auch $\int_0^L \sin^2(kx_i) = \frac{L}{2} - \frac{\sin(2kL)}{4k} \overset{(*)}{=} \frac{L}{2} - \frac{\sin(2\pi n_{x_i})}{4k} = \frac{L}{2}$ $\frac{L}{2}$!

$$\Rightarrow 1 = 64|A|^6 \frac{L^3}{8} \Leftrightarrow |A|^6 = \frac{1}{8L^3} \Leftrightarrow |A|^2 = \frac{1}{2L} \Leftrightarrow |A| = \sqrt{\frac{1}{2L}}$$

Das i hat da eh nix verloren :)

Um das i aus dem Ψ zu bekommen darf man zusätzlich mit i normieren, da dies den Betrag nicht ändert.

$$\Rightarrow A = -\sqrt{\frac{1}{2L}}\, i \Rightarrow A^3 = +\sqrt{\frac{1}{8L^3}}\, i$$

(*) Quantisierungsbedingung

$$\Rightarrow \Psi(\vec r) \overset{(*)}{=} 8\sqrt{\frac{1}{8L^3}} \sin\left(\frac{n_x \pi}{L}\right) \sin\left(\frac{n_y \pi}{L}\right) \sin\left(\frac{n_z \pi}{L}\right)$$

$$= \sqrt{\frac{8}{L^3}} \sin\left(\frac{n_x \pi}{L}\right) \sin\left(\frac{n_y \pi}{L}\right) \sin\left(\frac{n_z \pi}{L}\right) (\checkmark)$$

Einsetzen in die Schrödingergleichung liefert:

$$-\frac{\hbar^2}{2m}\nabla^2 \Psi(\vec r) = E\Psi(\vec r), \text{ da } V(\vec r) = 0 \text{ in unserem Bereich}$$

$$\Rightarrow -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\Psi(\vec r) = E\Psi(\vec r)$$

Es gilt: $\frac{\partial^2}{\partial x_i^2}\Psi(\vec r) = \left(\frac{\partial^2}{\partial x_i^2}\phi_{x_i}\right)\phi_{x_j}\phi_{x_k} = -\frac{2mE}{\hbar^2}\phi_{x_i}\phi_{x_j}\phi_{x_k}$

$$\overset{\text{Quant.-bed.}}{=} -\frac{n_{x_i}^2\pi^2}{L^2}\phi_{x_i}\phi_{x_j}\phi_{x_k}$$

$$\Rightarrow -\frac{\hbar^2}{2m}\left(-\frac{n_x^2\pi^2}{L^2} - \frac{n_y^2\pi^2}{L^2} - \frac{n_z^2\pi^2}{L^2}\right)\phi_x\phi_y\phi_z = E\Psi(\vec r)$$

$$\Leftrightarrow +\frac{\hbar^2}{2m} \cdot \frac{\pi^2}{L^2}\left(n_x^2 + n_y^2 + n_z^2\right) = E = E_x + E_y + E_z$$

Die Anzahl der Energiewerte ist unendlich, da n_{x_i} beliebig, natürlich.

Skizze: ✓

$\phi_x = 2iA \sin\left(\sqrt{\frac{2mE}{\hbar^2}}\, x\right)$, wobei A zu $A = -\sqrt{\frac{1}{2L}}\, i$ ausgerechnet wurde.

$$\Rightarrow \phi_x = 2\sqrt{\frac{1}{2L}} \sin\left(\sqrt{\frac{2mE}{\hbar^2}}\, x\right) = \sqrt{\frac{2}{L}} \sin\left(\sqrt{\frac{2mE}{\hbar^2}}\, x\right)$$

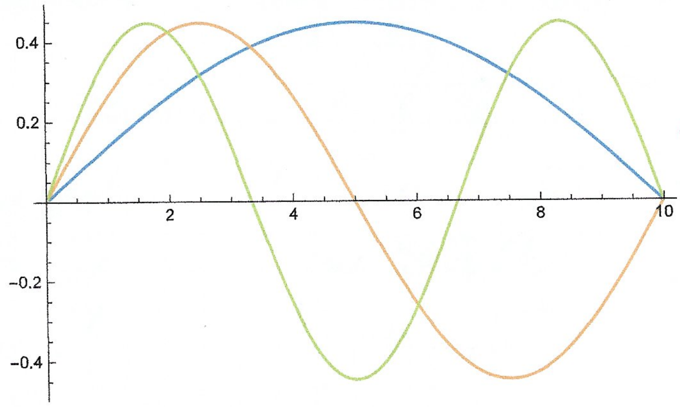
Und $|\phi_x(x)|^2 = \frac{2}{L} \sin^2\left(\sqrt{\frac{2mE}{\hbar^2}}\, x\right)$ ✓ Hier stimmt die Normierung auf einmal ... oder ich habe eine 2 übersehen ;)

mit Quantisierungsbed. gilt: $\sqrt{\frac{2mE}{\hbar^2}}\, L = n\pi$ und damit

$$\phi_x(x) = \sqrt{\frac{2}{L}} \sin\left(n_x\pi/L\; x\right), \quad |\phi_x(x)|^2 = \frac{2}{L} \sin^2\left(\frac{n\pi}{L} x\right)$$

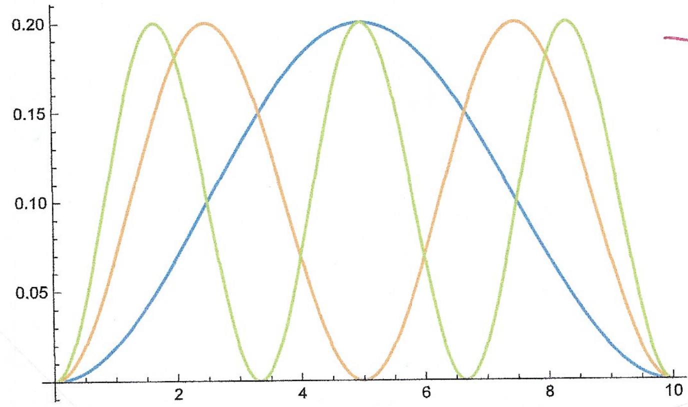
```
In[1]:= f[x_, n_] := Sqrt[2 / L] * Sin[n * Pi / L * x]

In[2]:= fab[z_, m_] := Abs[f[z, m]] ^2

In[9]:= L := 10;
        Plot[{f[x, 1], f[x, 2], f[x, 3]}, {x, 0, L}]
```

Out[10]=




```
In[11]:= Plot[{fab[x, 1], fab[x, 2], fab[x, 3]}, {x, 0, L}]
```

Out[11]=



Sehr gut! Hat sonst kaum wer bedacht...

2. $L = 2nm$, $m_{eff} = 0{,}2 m_e$

$E = \frac{\hbar^2}{2m} \frac{\pi^2}{L^2}(n_x^2 + n_y^2 + n_z^2)$, $C = \frac{\hbar^2}{2m} \frac{\pi^2}{L^2} = 0{,}47 eV$

$E_{111} = 3C = 1{,}41 eV$

$6C = E_{112} = E_{121} = E_{211} = 2{,}82 eV$

$12C = E_{222} = 5{,}64 eV$

$E_{223} = E_{232} = E_{322} = 17C = 7{,}99 eV$

$E_{333} = 27C = 12{,}69 eV$

$E_{122} = E_{221} = E_{212} = 4{,}23 eV$

$E_{233} = E_{323} = E_{332} = 10{,}34 eV$

Nehmen wir an, dass das Elektron immer wieder in den Grundzustand fällt. Dann werden folgende Wellenlängen emittiert/absorbiert (falls es sich vorher im Grundzustand befindet):

$\lambda = \frac{c}{f} = \frac{c}{\Delta E} \hbar \cdot 2\pi$ und $\Delta E = (E_i - E_j) \cdot e$ wenn E_i, E_j in eV

$\lambda_{333 \to 111} = 109{,}9 nm$

$\lambda_{233 \to 111} = 138{,}84 nm$

$\lambda_{122 \to 111} = 439{,}66 \cdot 10^{-7}$

$\lambda_{223 \to 111} = 188{,}42 nm$

$\lambda_{222 \to 111} = 293{,}1 nm$

$\lambda_{112 \to 111} = 879{,}32 nm$

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II) $[\hat{x}, \hat{p}_x] = (\hat{x}\hat{p}_x - \hat{p}_x\hat{x})\psi = x(-i\hbar\,\partial_x)\psi - (-i\hbar\,\partial_x)(x\psi)$

$\boxed{1}$

$$= -i\hbar x\,\partial_x\psi + i\hbar(\psi + x\,\partial_x\psi)$$

$$= i\hbar\psi \Rightarrow [\hat{x}, \hat{p}_x] = i\hbar$$

$$[\hat{x}, \hat{p}_y] = (\hat{x}\hat{p}_y - \hat{p}_y\hat{x})\psi = x(-i\hbar\,\partial_y)\psi - (-i\hbar\,\partial_y)(x\psi)$$

$$= -i\hbar x\,\partial_y\psi + i\hbar x\,\partial_y\psi = 0$$

$$[\hat{x}, \hat{p}_z] = (\hat{x}\hat{p}_z - \hat{p}_z\hat{x})\psi = x(-i\hbar\,\partial_z)\psi - (-i\hbar\,\partial_z)(x\psi)$$

$$= -i\hbar x\,\partial_z\psi + i\hbar x\,\partial_z\psi = 0$$

Ortsunschärfe in x-Richtung und Impulsunschärfe in x-Richtung sind miteinander gekoppelt, allerdings ist eine Ortsbestimmung in x-Richtung unabhängig von der y, z-Komponente des Impuls.

$\boxed{2}$ $\hat{H} = \frac{\hat{p}^2}{2m} + V(\vec{x})$, wobei $\hat{p}^2 = \hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2$

$$[\hat{H}, \hat{x}] = \left(\frac{\hat{p}^2}{2m} + V(\vec{x})\right)(\hat{x}\psi) - \hat{x}\left(\frac{\hat{p}^2}{2m} + V(\vec{x})\right)\psi$$

$$= \left(\frac{\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2}{2m}\right)(x\psi) - x\left(\frac{\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2}{2m}\right)\psi$$

$$+ V(\vec{x})(x\psi) - xV(\vec{x})\psi$$

$$= \frac{-\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(x\psi) + x\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\psi$$

$$= \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right)(x\psi) + x\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi$$

$$= \frac{\hbar^2}{2m}\left(x\frac{\partial^2}{\partial x^2}\psi - \partial_x(\psi + x\,\partial_x\psi)\right)$$

$$= \frac{\hbar^2}{2m}\left(x\frac{\partial^2}{\partial x^2}\psi - \frac{\partial}{\partial x}\psi - x\frac{\partial^2}{\partial x^2}\psi - \frac{\partial}{\partial x}\psi\right)$$

$$= \frac{\hbar^2}{2m}(-2\,\partial_x\psi) = -\frac{\hbar^2}{m}\partial_x\psi \Rightarrow [\hat{H}, \hat{x}] = -\frac{\hbar^2}{m}\partial_x$$

$$[\hat{H}, \hat{p}_x] = \left(\frac{\hat{p}^2}{2m} + V(\vec{x})\right)(-i\hbar\,\partial_x)\psi - (-i\hbar\,\partial_x)\left(\frac{\hat{p}^2}{2m} + V(\vec{x})\right)\psi$$

$$= \left(\frac{\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2}{2m}\right)(-i\hbar\,\partial_x)\psi + (i\hbar\,\partial_x)\left(\frac{\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2}{2m}\right)\psi$$

$$+ V(\vec{x})(-i\hbar\,\partial_x)\psi + (i\hbar\,\partial_x)V(\vec{x})\psi$$

$$= \frac{-\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(-i\hbar\,\partial_x)\psi + (i\hbar\,\partial_x)\frac{-\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\psi$$

$$+ V(\vec{x})(-i\hbar\,\partial_x)\psi + (i\hbar\,\partial_x)V(\vec{x})\psi$$

$$= -i\hbar V(\vec{x})\,\partial_x\psi + i\hbar\left[(\partial_x V(\vec{x}))\psi + V(\vec{x})\,\partial_x\psi\right] = i\hbar\left(\frac{\partial}{\partial x}V(\vec{x})\right)\psi$$

$$\Rightarrow [\hat{H}, \hat{p}_x] = i\hbar\left(\frac{\partial}{\partial x}V(\vec{x})\right)$$

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$$\hat{L} = \hat{x} \times \hat{p} = \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{pmatrix} \times \begin{pmatrix} \hat{p}_1 \\ \hat{p}_2 \\ \hat{p}_3 \end{pmatrix} = \begin{pmatrix} \hat{x}_2 \hat{p}_3 - \hat{x}_3 \hat{p}_2 \\ \hat{x}_3 \hat{p}_1 - \hat{x}_1 \hat{p}_3 \\ \hat{x}_1 \hat{p}_2 - \hat{x}_2 \hat{p}_1 \end{pmatrix} = \begin{pmatrix} -y i\hbar \frac{\partial}{\partial z} + z i\hbar \frac{\partial}{\partial y} \\ -z i\hbar \frac{\partial}{\partial x} + x i\hbar \frac{\partial}{\partial z} \\ -x i\hbar \frac{\partial}{\partial y} + y i\hbar \frac{\partial}{\partial x} \end{pmatrix}$$

$$= \begin{pmatrix} -i\hbar \left(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y}\right) \\ -i\hbar \left(z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z}\right) \\ -i\hbar \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}\right) \end{pmatrix}$$

$$[\hat{L}_z, \hat{L}_y] = -\hbar^2 \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}\right)\left(z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z}\right)\psi + \hbar^2 \left(z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z}\right)\left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}\right)\psi$$

$$= -\hbar^2 \left(xz \frac{\partial^2}{\partial y \partial x} - x^2 \frac{\partial^2}{\partial y \partial z} - yz \frac{\partial^2}{\partial x^2} + y \frac{\partial}{\partial z} + yx \frac{\partial^2}{\partial x \partial z}\right)\psi$$

$$+ \hbar^2 \left(z \frac{\partial}{\partial y} + zx \frac{\partial^2}{\partial y \partial x} - zy \frac{\partial^2}{\partial x^2} - x^2 \frac{\partial^2}{\partial z \partial y} + xy \frac{\partial^2}{\partial z \partial x}\right)\psi$$

$$= \hbar^2 \left(z \frac{\partial}{\partial y} - y \frac{\partial}{\partial z}\right)\psi = \frac{\hat{L}_x}{i} \hbar \psi = -i\hbar \hat{L}_x \psi \Rightarrow [\hat{L}_z, \hat{L}_y] = -i\hbar \hat{L}_x$$ ✓

$$[\hat{L}_z, \hat{z}] = -i\hbar \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}\right)(z\psi) - z\left(-i\hbar\left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}\right)\right)\psi$$

$$= -i\hbar z \left(x \frac{\partial}{\partial y}\psi - y \frac{\partial}{\partial x}\psi\right) + i\hbar z \left(x \frac{\partial}{\partial y}\psi - y \frac{\partial}{\partial x}\psi\right) = 0$$ ✓

$$[\hat{L}_z, \hat{p}_z] = -i\hbar \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}\right)\left(-i\hbar \frac{\partial}{\partial z}\right)\psi - \left(-i\hbar \frac{\partial}{\partial z}\right)\left(-i\hbar\left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}\right)\right)\psi$$

$$= -\hbar^2 \left(x \frac{\partial^2}{\partial y \partial z}\psi - y \frac{\partial^2}{\partial x \partial z}\psi\right) + \hbar^2 \left(x \frac{\partial^2}{\partial z \partial y}\psi - y \frac{\partial^2}{\partial z \partial x}\psi\right) = 0$$ ✓

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III) $[\hat{p}^2, \hat{L}_z] = [\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2, \hat{L}_z] = [\hat{p}_x^2, \hat{L}_z] + [\hat{p}_y^2, \hat{L}_z] + [\hat{p}_z^2, \hat{L}_z]$

- $[\hat{p}_x^2, \hat{L}_z] = -\hbar^2 \frac{\partial^2}{\partial x^2}\left(-i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right)\psi - \left(-i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right)\left(-\hbar^2\frac{\partial^2}{\partial x^2}\right)\psi$

$= i\hbar^3 \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} + x\frac{\partial^2}{\partial x\partial y} - y\frac{\partial^2}{\partial x^2}\right)\psi - i\hbar^3\left(x\frac{\partial^3}{\partial y\partial x^2} - y\frac{\partial^3}{\partial x^3}\right)$

$= i\hbar^3\left(\frac{\partial^2}{\partial x\partial y} + \frac{\partial^2}{\partial x\partial y} + x\frac{\partial^3}{\partial x^2\partial y} - y\frac{\partial^3}{\partial x^3}\right)\psi - i\hbar^3\left(x\frac{\partial^3}{\partial y\partial x^2} - y\frac{\partial^3}{\partial x^3}\right)\psi$

$= 2i\hbar^3 \frac{\partial^2}{\partial x\partial y}$

- $[\hat{p}_y^2, \hat{L}_z] = -\hbar^2\frac{\partial^2}{\partial y^2}\left(-i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right)\psi - \left(-i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right)\left(-\hbar^2\frac{\partial^2}{\partial y^2}\right)\psi$

$= i\hbar^3\left(x\frac{\partial^3}{\partial y^3}\right) + i\hbar^3\frac{\partial}{\partial y}\left(-\frac{\partial}{\partial x} - y\frac{\partial^2}{\partial x\partial y}\right)\psi - i\hbar^3\left(x\frac{\partial^3}{\partial y^3} - y\frac{\partial^3}{\partial x\partial y^2}\right)\psi$

$= i\hbar^3\left(-\frac{\partial^2}{\partial x\partial y} - \frac{\partial^2}{\partial x\partial y} - y\frac{\partial^3}{\partial x\partial y^2}\right)\psi - i\hbar^3\left(-y\frac{\partial^3}{\partial x\partial y^2}\right)\psi$

$= -2i\hbar^3\left(\frac{\partial^2}{\partial x\partial y}\right)$

- $[\hat{p}_z^2, \hat{L}_z] = -\hbar^2\frac{\partial^2}{\partial z^2}\left(-i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right)\psi - \left(-i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right)\left(-\hbar^2\frac{\partial^2}{\partial z^2}\right)\psi$

$= i\hbar^3\frac{\partial^2}{\partial z^2}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\psi - i\hbar^3\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\frac{\partial^2}{\partial z^2}\psi$

$= 0$

$\Rightarrow [\hat{p}^2, \hat{L}_z] = 0$

$[\hat{r}^2, \hat{L}_z] = [\hat{x}^2, \hat{L}_z] + [\hat{y}^2, \hat{L}_z] + [\hat{z}^2, \hat{L}_z]$

- $[\hat{x}^2, \hat{L}_z] = \hat{x}^2\left(-i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right)\psi - \left(-i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right)\hat{x}^2$

$= \left(-i\hbar x^3\frac{\partial}{\partial y} + i\hbar y x^2\frac{\partial}{\partial x}\right)\psi + i\hbar x^3\frac{\partial}{\partial y}\psi - i\hbar y\left(2x\psi + x^2\psi\right)$

$= i\hbar y x^2\frac{\partial}{\partial x}\psi - 2i\hbar xy\psi - i\hbar y x^2\psi = -2i\hbar xy\psi$

- $[\hat{y}^2, \hat{L}_z] = \hat{y}^2\left(-i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right)\psi - \left(-i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right)\hat{y}^2\psi$

$= -i\hbar y^2 x\frac{\partial}{\partial y}\psi + i\hbar y^3\frac{\partial}{\partial x}\psi + i\hbar\left(x\left(2y\psi + y^2\frac{\partial}{\partial y}\psi\right) - y^3\frac{\partial}{\partial x}\psi\right)$

$= -i\hbar y^2 x\frac{\partial}{\partial y}\psi + 2i\hbar xy\psi + i\hbar y^2\frac{\partial}{\partial y}\psi = 2i\hbar xy\psi$

- $[\hat{z}^2, \hat{L}_z] = \hat{z}^2\left(-i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right)\psi - \left(-i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right)\hat{z}^2\psi$

$= -i\hbar xz^2\frac{\partial}{\partial y}\psi + i\hbar yz^2\frac{\partial}{\partial x}\psi + i\hbar xz^2\frac{\partial}{\partial y}\psi - i\hbar yz^2\frac{\partial}{\partial x}\psi = 0$

$\Rightarrow [\hat{r}^2, \hat{L}_z] = 0$

$[f(\hat{r}^2), \hat{L}_z] = f(\hat{x}^2 + \hat{y}^2 + \hat{z}^2)\hat{L}_z - \hat{L}_z f(\hat{x}^2 + \hat{y}^2 + \hat{z}^2)$

$\hat{r}^2 = \hat{x}^2 + \hat{y}^2 + \hat{z}^2$

$= f(\hat{x}^2 + \hat{y}^2 + \hat{z}^2)\left(-i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right)\psi - \left(-i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right)f(\hat{x}^2 + \hat{y}^2 + \hat{z}^2)\psi$

$= -i\hbar f(\hat{r}^2)\left(x\frac{\partial}{\partial y}\psi - y\frac{\partial}{\partial x}\psi\right) + i\hbar x\frac{\partial f}{\partial \hat{r}^2}\cdot 2y\psi - i\hbar y\frac{\partial f}{\partial \hat{r}^2}\cdot 2x\psi$

$+ i\hbar x f\frac{\partial}{\partial y}\psi \quad - i\hbar y f\frac{\partial}{\partial x}\psi$

$= 2i\hbar xy\left[\frac{\partial f(\hat{r}^2)}{\partial \hat{r}^2}\psi - \frac{\partial f(\hat{r}^2)}{\partial \hat{r}^2}\psi\right] = 0$

Kommentar?

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$\boxed{\text{IV}}$ $V = \frac{1}{2} m \omega^2 x^2$

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}}, \quad \psi_1(x) = \frac{\sqrt{2}}{\pi^{1/4}} x \left(\frac{m\omega}{\hbar}\right)^{3/4} e^{-\frac{m\omega x^2}{2\hbar}}$$

Der Zeitentwicklungsoperator $U(t) = e^{-i\frac{E}{\hbar}t}$ ✓ ergibt nach Anwendung die zeitabhängige ~~Sch~~ Lösung der Schrödingerglg.

$$\psi_0(x,t) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar} - i\frac{\omega}{2}t} \qquad \text{da } E_0 = \hbar\omega/2 \checkmark$$

$$\psi_1(x,t) = \frac{\sqrt{2}}{\pi^{1/4}} x \left(\frac{m\omega}{\hbar}\right)^{3/4} e^{-\frac{m\omega x^2}{2\hbar} - i\frac{3\omega}{2}t}, \quad E_1 = \frac{3}{2}\hbar\omega \checkmark$$

$\Rightarrow$ Lösen beide die zeitabhängige Schrödingerglg.:

$$\frac{-\hbar^2}{2m}\nabla^2\psi_i(x,t) + \frac{1}{2}m\omega^2x^2\psi_i(x,t) = i\hbar\frac{\partial}{\partial t}\psi_i(x,t)$$

Und damit auch die Summe (+ skalarer Vorfaktor)

$$\Psi(x,t) = \frac{1}{\sqrt{2}}\left(\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar} - i\frac{\omega}{2}t} + \frac{\sqrt{2}}{\pi^{1/4}} x \left(\frac{m\omega}{\hbar}\right)^{3/4} e^{-\frac{m\omega x^2}{2\hbar} - i\frac{3}{2}\omega t}\right)$$

$$|\Psi(x,t)|^2 = \frac{1}{2}\left(\left(\frac{m\omega}{\pi\hbar}\right)^{1/2} e^{-\frac{m\omega x^2}{\hbar}} + \frac{2}{\pi^{1/2}} x^2 \left(\frac{m\omega}{\hbar}\right)^{3/2} e^{-\frac{m\omega x^2}{\hbar}}\right.$$

$$+ \left(\frac{m\omega}{\pi\hbar}\right) x \frac{\sqrt{2}}{\pi^{1/4}} e^{-\frac{m\omega x^2}{\hbar} + i\omega t}$$

$$+ \frac{\sqrt{2}}{\pi^{1/4}} x \left(\frac{m\omega}{\hbar}\right) e^{-\frac{m\omega x^2}{\hbar} - i\omega t}$$

$$\left(\frac{m\omega}{\pi\hbar}\right)\frac{\sqrt{2}}{\pi^{1/4}} x e^{-\frac{m\omega x^2}{\hbar}} \underbrace{\left(e^{i\omega t} + e^{-i\omega t}\right)}_{\cos(\omega t)}$$

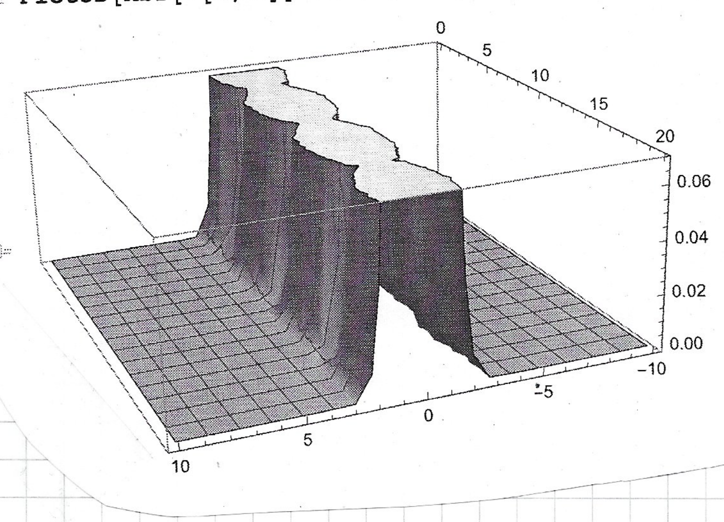
$$\frac{1}{2}\left(\left(\frac{m\omega}{\pi\hbar}\right)^{1/2} e^{-\frac{m\omega x^2}{\hbar}} + \frac{2}{\pi^{1/2}} x^2 \left(\frac{m\omega}{\hbar}\right)^{3/2} e^{-\frac{m\omega x^2}{\hbar}} \checkmark + \frac{m\omega}{\pi\hbar}\frac{\sqrt{2}}{\pi^{1/4}} x e^{-\frac{m\omega x^2}{\hbar}} \cos(\omega t)\right)$$

```
In[33]:= w := 1;
m := 1;
h := 1;
f[x_, t_] :=
 1 / Sqrt[2] * (((m * w) / (Pi * h)) ^ (1 / 2) * Exp[- (m * w * x ^ 2) / (2 * h) - i * w / 2 * t] +
   ((Sqrt[2]) / (Pi ^ (1 / 4)) * (m * w) / (h)) ^ (3 / 4) *
    Exp[- (m * w * x ^ 2) / (2 * h) - i * 3 / 2 * w * t])

In[39]:= Plot3D[Abs[f[a, b]] ^ 2, {a, -10, 10}, {b, 0, 20}]
```

Da lief was schief...

Out[39]=



Wie soll ich denn $\Psi(x,t)$ plotten? Das wäre ja 4D... - $\Psi(x)$ - $\Psi(t)$ (2D!) :)

$$\frac{d}{dt}\langle\hat{x}\rangle = \frac{i}{\hbar}\langle\underbrace{[\hat{H},\hat{x}]}_{-\frac{\hbar^2}{2m}\partial/\partial x \text{ nach Aufgabe 2.2.}}\rangle + \langle\underbrace{\frac{\partial x}{\partial t}}_{0}\rangle$$

$$= -\frac{i\hbar}{2m}\langle \partial/\partial x\, \psi\rangle$$ $\left[\langle x\rangle = \int |\psi|^2 x\, dx = \sqrt{\frac{\hbar}{2m\omega}}\cos(\omega t)\right.$

2. $$\frac{d}{dt}\langle\hat{p}\rangle = \frac{i}{\hbar}\langle[\hat{H},\hat{p}]\rangle + \langle\underbrace{\frac{\partial\hat{p}}{\partial t}}_{0}\rangle$$

$$= \frac{i}{\hbar}\langle i\hbar\left(\partial/\partial x\, V + \partial/\partial y\, V + \partial/\partial z\, V\right)\rangle$$

$$= -\langle \partial/\partial x\, V + \partial/\partial y\, V + \partial/\partial z\, V\rangle = -\langle \operatorname{div} V\rangle$$ $\left[\langle p\rangle(t) = m\cdot\langle\dot{x}\rangle = -\sqrt{\frac{m\omega\hbar}{2}}\sin(\omega t)\right.$

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