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General Relativity Sheet 1 — Marvin Zanke

good 13/15

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H1)

1

$\eta_{\mu\nu} = \epsilon_{(\mu)} \delta_{\mu\nu}$ w/o summation

Y spherical polar coordinates, $y^0 = x^0$, $y^1 = r$, $y^2 = \theta$, $y^3 = \phi$

$x^1 = r\sin\theta\cos\phi$, $x^2 = r\sin\theta\sin\phi$, $x^3 = r\cos\theta$

Where from this definition of $g_{\mu\nu}(y)$? Not the trafo of a (2,0)-Tensor-field? Why use $\frac{\partial x^\alpha}{\partial y^\mu}$ to transform to different coordinates? → $g_{\mu\nu}(y)$ is still the flat space-time! No curvature here.

a) $$g_{\mu\nu}(y) = \eta_{\alpha\beta} \frac{\partial x^\alpha}{\partial y^\mu} \frac{\partial x^\beta}{\partial y^\nu}$$

Careful with index position. should be $\underline{\underline{M^\alpha{}_\mu}}$.

We calculate $\frac{\partial x^\alpha}{\partial y^\mu} \equiv M_{\alpha\mu}$ first

$\frac{\partial x^0}{\partial y^0} = 1$, $\frac{\partial x^0}{\partial y^1} = 0$, $\frac{\partial x^0}{\partial y^2} = 0$, $\frac{\partial x^0}{\partial y^3} = 0$

$\frac{\partial x^1}{\partial y^0} = 0$, $\frac{\partial x^1}{\partial y^1} = \sin\theta\cos\phi$, $\frac{\partial x^1}{\partial y^2} = r\cos\theta\cos\phi$, $\frac{\partial x^1}{\partial y^3} = -r\sin\theta\sin\phi$

$\frac{\partial x^2}{\partial y^0} = 0$, $\frac{\partial x^2}{\partial y^1} = \sin\theta\sin\phi$, $\frac{\partial x^2}{\partial y^2} = r\cos\theta\sin\phi$, $\frac{\partial x^2}{\partial y^3} = r\sin\theta\cos\phi$

$\frac{\partial x^3}{\partial y^0} = 0$, $\frac{\partial x^3}{\partial y^1} = \cos\theta$, $\frac{\partial x^3}{\partial y^2} = -r\sin\theta$, $\frac{\partial x^3}{\partial y^3} = 0$

and then notice: $$g_{\mu\nu}(y) = \sum_{\alpha\beta} \epsilon_{(\alpha)} \delta_{\alpha\beta} \frac{\partial x^\alpha}{\partial y^\mu} \frac{\partial x^\beta}{\partial x^\nu} = \sum_\alpha \epsilon_\alpha \frac{\partial x^\alpha}{\partial y^\mu} \frac{\partial x^\alpha}{\partial x^\nu} \quad (*)$$

and alternatively $$g_{\mu\nu}(y) = \frac{\partial x^\alpha}{\partial y^\mu} \eta_{\alpha\beta} \frac{\partial x^\beta}{\partial y^\nu} = M_{\alpha\mu} \eta_{\alpha\beta} M_{\beta\nu}$$

$$= (M^T)_{\mu\alpha} \eta_{\alpha\beta} M_{\beta\nu} = (M^T \eta M)_{\mu\nu}$$

Same as before, Careful with index structure

where $$\eta_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$M_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ 0 & \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ 0 & \cos\theta & -r\sin\theta & 0 \end{pmatrix}$$

We find

$$g_{\mu\nu}(y) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix}$$

using $\cos^2 + \sin^2 = 1$, as we calculate using the first eq. (*)

- $g_{00}(y) = 1$, $g_{11}(y) = -\sin^2\theta(\cos^2\phi + \sin^2\phi) - \cos^2\theta = -1$

$g_{22}(y) = -r^2\cos^2\theta(\cos^2\phi + \sin^2\phi) - r^2\sin^2\theta = -r^2$

$g_{33}(y) = -r^2\sin^2\theta(\sin^2\phi + \cos^2\phi) = -r^2\sin^2\theta$

- $g_{0i}(y) = 0$ is trivial

$g_{12}(y) = -r\cos^2\phi(\sin\theta\cos\theta) - r\sin^2\phi(\sin\theta\cos\theta) + r\sin\theta\cos\theta = 0$

$g_{13}(y) = +r\sin^2\theta\sin\phi\cos\phi - r\sin^2\theta\sin\phi\cos\phi = 0$

$g_{23}(y) = -r^2\sin\theta\cos\theta\sin\phi\cos\phi + r^2\sin\theta\cos\theta\sin\phi\cos\phi = 0$

$+ \ g_{\mu\nu} = g_{\nu\mu}$ symmetric

With $g = J^T \eta J$ not really more elegant but is there a nicer way $\rightsquigarrow$ no shortcut way, as we want to ch

b) Have $\vec{E} = Q\dfrac{\vec{x}}{|\vec{x}|^3}$, $\vec{B} = 0$

Per definition, we have $F_{00} = 0$, $F_{0i} = -F_{i0} = E^i$, $F_{ik} = -\epsilon_{ik\ell}B^\ell \underset{\vec{B}=0}{=} 0$

1

$$\rightsquigarrow F_{\mu\nu} = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & 0 & 0 \\ -E^2 & 0 & 0 & 0 \\ -E^3 & 0 & 0 & 0 \end{pmatrix} = \frac{Q}{|\vec{x}|^3}\begin{pmatrix} 0 & x^1 & x^2 & x^3 \\ -x^1 & 0 & 0 & 0 \\ -x^2 & 0 & 0 & 0 \\ -x^3 & 0 & 0 & 0 \end{pmatrix}$$

c) $\hat{F}_{\mu\nu}(y) = F_{\alpha\beta}(y)\dfrac{\partial x^\alpha}{\partial y^\mu}\dfrac{\partial x^\beta}{\partial y^\nu}$ obviously antisym. under $\mu \leftrightarrow \nu$ as $F_{\alpha\beta}$ antisym. under $\alpha \leftrightarrow \beta$

1

We notice, that only for $(\alpha,\beta) = (i,0)$ or $(0,i)$ there's a contribution. Looking at $\dfrac{\partial x^\alpha}{\partial y^\mu}\dfrac{\partial x^\beta}{\partial y^\nu}$ we see that then, if $\alpha = 0$, we need $\mu = 0$ for a non-vanishing contribution, if $\beta = 0$, need $\nu = 0$, while $\alpha = i \Rightarrow \mu \neq 0$, $\beta = i \Rightarrow \nu \neq 0$. We find $\hat{F}_{\mu\nu}(y) = \begin{pmatrix} 0 & F_{01} & F_{02} & F_{03} \\ F_{10} & 0 & 0 & 0 \\ F_{20} & 0 & 0 & 0 \\ F_{30} & 0 & 0 & 0 \end{pmatrix}$

- $\hat{F}_{01}(y) = F_{\alpha\beta}(y)\dfrac{\partial x^\alpha}{\partial y^0}\dfrac{\partial x^\beta}{\partial y^1} = \dfrac{\partial x^0}{\partial y^0}\left\{F_{01}(y)\dfrac{\partial x^1}{\partial y^1} + F_{02}(y)\dfrac{\partial x^2}{\partial y^1} + F_{03}(y)\dfrac{\partial x^3}{\partial y^1}\right\}$

$= \dfrac{Q}{r^3}\left\{r\sin^2\theta\cos^2\phi + r\sin^2\theta\sin^2\phi + r\cos^2\theta\right\} = \dfrac{Q}{r^2}$

$\hat{F}_{02}(y) = \dfrac{\partial x^0}{\partial y^0}\left\{F_{01}(y)\dfrac{\partial x^1}{\partial y^2} + F_{02}(y)\dfrac{\partial x^2}{\partial y^2} + F_{03}(y)\dfrac{\partial x^3}{\partial y^2}\right\}$

$= \dfrac{Q}{r^3}\left\{r^2\cos^2\phi\sin\theta\cos\theta + r^2\sin^2\phi\sin\theta\cos\theta - r^2\sin\theta\cos\theta\right\} = 0$

$\hat{F}_{03}(y) = \dfrac{\partial x^0}{\partial y^0}\left\{F_{01}(y)\dfrac{\partial x^1}{\partial y^3} + F_{02}(y)\dfrac{\partial x^2}{\partial y^3} + F_{03}(y)\dfrac{\partial x^3}{\partial y^3}\right\}$

$= \dfrac{Q}{r^3}\left\{-r^2\sin^2\theta\sin\phi\cos\phi + r^2\sin^2\theta\sin\phi\cos\phi\right\} = 0$

GOOD

$$\hat{F}_{\mu\nu}(y) = \begin{pmatrix} 0 & Q/r^2 & 0 & 0 \\ -Q/r^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

H2)

$$\Gamma^{\gamma}_{\beta\alpha} = \frac{1}{2} g^{\gamma\delta} \left(\frac{\partial}{\partial x^\beta} g_{\delta\alpha} + \frac{\partial}{\partial x^\alpha} g_{\delta\beta} - \frac{\partial}{\partial x^\delta} g_{\alpha\beta} \right)$$

a)

$$g_{\alpha\beta;\mu} = \frac{\partial}{\partial x^\mu} g_{\alpha\beta} - g_{\kappa\beta}\Gamma^{\kappa}_{\alpha\mu} - g_{\alpha\kappa}\Gamma^{\kappa}_{\beta\mu}$$

$$= \frac{\partial}{\partial x^\mu} g_{\alpha\beta} - g_{\kappa\beta}\left\{\frac{1}{2} g^{\kappa\delta}\left(\frac{\partial}{\partial x^\alpha} g_{\delta\mu} + \frac{\partial}{\partial x^\mu} g_{\delta\alpha} - \frac{\partial}{\partial x^\delta} g_{\alpha\mu}\right)\right\} - g_{\alpha\kappa}\left\{\frac{1}{2} g^{\kappa\delta}\left(\frac{\partial}{\partial x^\beta} g_{\delta\mu} + \frac{\partial}{\partial x^\mu} g_{\delta\beta} - \frac{\partial}{\partial x^\delta} g_{\beta\mu}\right)\right\}$$

Can pull the δ^δ_β behind $\frac{\partial}{\partial x}$ without problems to apply on $g_{\delta\mu}$?
yes, no x-dep. $\delta^\delta_\beta \partial_\alpha g_{\delta\mu} = \partial_\alpha g_{\beta\mu}$ as only $\neq 0$ if $\delta = \beta$

$$\overset{g\text{ sym.}}{=} \frac{\partial}{\partial x^\mu} g_{\alpha\beta} - \frac{1}{2}\left\{ \delta^\delta_\beta \left(\frac{\partial}{\partial x^\alpha} g_{\delta\mu} + \frac{\partial}{\partial x^\mu} g_{\delta\alpha} - \frac{\partial}{\partial x^\delta} g_{\alpha\mu}\right) + \delta^\delta_\alpha \left(\frac{\partial}{\partial x^\beta} g_{\delta\mu} + \frac{\partial}{\partial x^\mu} g_{\delta\beta} - \frac{\partial}{\partial x^\delta} g_{\beta\mu}\right)\right\}$$

$$= \frac{\partial}{\partial x^\mu} g_{\alpha\beta} - \frac{1}{2}\left\{ \frac{\partial}{\partial x^\alpha} g_{\beta\mu} + \frac{\partial}{\partial x^\mu} g_{\beta\alpha} - \frac{\partial}{\partial x^\beta} g_{\alpha\mu} + \frac{\partial}{\partial x^\beta} g_{\alpha\mu} + \frac{\partial}{\partial x^\mu} g_{\alpha\beta} - \frac{\partial}{\partial x^\alpha} g_{\beta\mu} \right\}$$

$$\overset{g\text{ sym.}}{=} \frac{\partial}{\partial x^\mu} g_{\alpha\beta} - \frac{\partial}{\partial x^\mu} g_{\alpha\beta} = 0$$

b) Conversely assume

$\nabla_\mu g_{\alpha\beta} = 0$ — metric compatibility / torsion free

$$\frac{\partial}{\partial x^\mu} g_{\alpha\beta} - g_{\lambda\beta}\Gamma^{\lambda}_{\alpha\mu} - g_{\alpha\lambda}\Gamma^{\lambda}_{\beta\mu} = 0 \quad \text{and} \quad \Gamma^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\gamma\beta} \qquad T^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma} - \Gamma^{\alpha}_{\gamma\beta}$$

which can be written down for different variables, yielding:

$$\frac{\partial}{\partial x^\beta} g_{\delta\alpha} - g_{\lambda\alpha}\Gamma^{\lambda}_{\delta\beta} - g_{\delta\lambda}\Gamma^{\lambda}_{\alpha\beta} = 0$$

$$\frac{\partial}{\partial x^\alpha} g_{\delta\beta} - g_{\lambda\beta}\Gamma^{\lambda}_{\delta\alpha} - g_{\delta\lambda}\Gamma^{\lambda}_{\beta\alpha} = 0 \qquad \Gamma^{\lambda}_{\eta\epsilon} = \Gamma^{\lambda}_{\epsilon\eta}$$

$$\frac{\partial}{\partial x^\delta} g_{\alpha\beta} - g_{\lambda\beta}\Gamma^{\lambda}_{\alpha\delta} - g_{\alpha\lambda}\Gamma^{\lambda}_{\beta\delta} = 0$$

and thus

$$\frac{1}{2} g^{\gamma\delta}\left\{\frac{\partial}{\partial x^\beta} g_{\delta\alpha} + \frac{\partial}{\partial x^\alpha} g_{\delta\beta} - \frac{\partial}{\partial x^\delta} g_{\alpha\beta}\right\}$$

$$= \frac{1}{2} g^{\gamma\delta}\left\{ g_{\lambda\alpha}\Gamma^{\lambda}_{\delta\beta} + g_{\delta\lambda}\Gamma^{\lambda}_{\alpha\beta} + g_{\lambda\beta}\Gamma^{\lambda}_{\delta\alpha} + g_{\delta\lambda}\Gamma^{\lambda}_{\beta\alpha} - g_{\lambda\beta}\Gamma^{\lambda}_{\alpha\delta} - g_{\alpha\lambda}\Gamma^{\lambda}_{\beta\delta} \right\}$$

$$\overset{g\text{ sym.},\ \Gamma\text{ sym.}}{=} \frac{1}{2}\left\{ 2 g^{\gamma\delta} g_{\delta\lambda}\Gamma^{\lambda}_{\alpha\beta} \right\} = \Gamma^{\gamma}_{\alpha\beta} = \Gamma^{\gamma}_{\beta\alpha}$$

→ It is unique because once its action is specified for a particular set of basis, its action is completely determined. We will discuss it in class. Remind me if possible.

H3)

$$\omega(h_{\sigma_1}, \dots, h_{\sigma_n}) = \epsilon_\sigma \omega(h_1, \dots, h_n)$$

$A: V \to V$ linear trafo; determinant given by

$$\omega(Ah_1, \dots, Ah_n) = \det[A]\, \omega(h_1, \dots, h_n)$$

Space of skew-sym. cov. tensors of rank n over a vector space of dimension n is one dimensional? $\leadsto \omega(e_1, \dots, e_n) = 1$; $\in \mathbb{R}$ $\Rightarrow \omega(e_{\sigma 1}, \dots, e_{\sigma n}) = \mathrm{sgn}$

a) Want to prove $\omega(Ah_1, h_2, \dots, h_n) + \dots + \omega(h_1, \dots, Ah_n)$

$$= \mathrm{Tr}[A]\, \omega(h_1, \dots, h_n),$$

where it is sufficient to prove it for a basis $h_\mu = e_\mu$,

because $\forall h_i \in V$, $h_i = c_i^\mu e_\mu$ and e.g.

? ω skew. sym.

$$\omega(c_1^\mu e_\mu, h_2, \dots, h_n) = \omega(c_1^\mu e_\mu, c_2^\mu e_\mu, \dots, c_n^\mu e_\mu)$$

More formal argument? $\leadsto$

$$= k\epsilon_\sigma \omega(c_1^1 e_1, c_2^2 e_2, \dots, c_n^n e_n) = c_i^i\, k \epsilon_\sigma \omega(e_1, \dots, e_n)$$

"product rule"

We then find:

When using Einstein summation, one index up, one down $\to a^{\mu k} e_k$

$$\omega(Ae_1, e_2, \dots, e_n) + \omega(e_1, Ae_2, \dots, e_n) + \dots + \omega(e_1, \dots, e_{n-1}, Ae_n)$$

?

$$= \omega(a_k^1 e_k, e_2, \dots, e_n) + \dots + \omega(e_1, \dots, e_{n-1}, a_k^n e_k)$$

(k columns, $\uparrow$ rows)

ω skew sym.

$$= \omega(a_1^1 e_1, e_2, \dots, e_n) + \dots + \omega(e_1, \dots, e_{n-1}, a_n^n e_n)$$

Only in Cartesian basis? $\leadsto$ Correct for all coordinates, e.g. ϱ, ϑ etc.

$$= \omega(e_1, e_2, \dots, e_n)\{a_1^1 + \dots + a_n^n\}$$

$$= \mathrm{Tr}[A]\, \omega(e_1, e_2, \dots, e_n)$$

2

Pull out numbers from tensor arguments (i.e. linear)? Yes

b) In order to calculate $\frac{\partial}{\partial \alpha} \det[A(\alpha)]$, we look at ($A^\alpha = A(\alpha)$)

$[(A^\alpha)^{-1}(A^\alpha)']$

$$\frac{\partial}{\partial \alpha} \omega(A^\alpha h_1, \dots, A^\alpha h_n) = \omega\left(\frac{\partial}{\partial \alpha} A^\alpha h_1, A^\alpha h_2, \dots, A^\alpha h_n\right) + \dots$$

$$+ \omega\left(A^\alpha h_1, A^\alpha h_2, \dots, \frac{\partial}{\partial \alpha} A^\alpha h_n\right)$$

2

$[(A^\alpha)^{-1}(A^\alpha)']$

GOOD

$$\overset{a)}{\underset{(\text{with } A^\alpha h_i = h_i')}{=}} \mathrm{Tr}\left[\left(\frac{\partial}{\partial \alpha} A^\alpha\right)(A^\alpha)^{-1}\right] \omega(A^\alpha h_1, \dots, A^\alpha h_n)$$

$$\underset{\text{def. of det.}}{\overset{\text{Tr cyclic}}{=}} \mathrm{Tr}\left[(A^\alpha)^{-1}\left(\frac{\partial}{\partial \alpha} A^\alpha\right)\right] \det[A^\alpha]\, \omega(h_1, \dots, h_n)$$

Also $\frac{\partial}{\partial\alpha} \omega(A^\alpha h_1, \dots, A^\alpha h_n) = \frac{\partial}{\partial\alpha}\left\{\det(A^\alpha)\, \omega(h_1, \dots, h_n)\right\}$

$= \left(\frac{\partial}{\partial\alpha} \det[A^\alpha]\right) \omega(h_1, \dots, h_n)$

which immediately implies $\frac{\partial}{\partial\alpha} \det[A(\alpha)] = \det[A(\alpha)] \operatorname{Tr}\left[A^{-1}(\alpha) \frac{\partial}{\partial\alpha} A(\alpha)\right]$

due to the one-dimensionality of the "ω-space".

c) $|g|(x) \equiv \det(g_{\alpha\beta}(x))$

1

answer!

$\frac{\partial}{\partial x^\mu} |g|(x) = \frac{\partial}{\partial x^\mu} \det(g_{\alpha\beta}(x))$

contravariant (0,2) tensor $g^{\kappa\alpha}$

$= \det(g_{\alpha\beta}(x)) \operatorname{Tr}\left[g^{-1}_{\alpha\beta} \frac{\partial}{\partial x^\mu} g_{\alpha\beta} \right]$

H1 generalized $\det(g_{\alpha\beta}(x)) \operatorname{Tr}\left[\eta^{\mu\nu} \frac{\partial y^\kappa}{\partial x^\mu} \frac{\partial y^\alpha}{\partial x^\nu} \frac{\partial}{\partial x^\mu} \eta_{\lambda\delta} \frac{\partial x^\lambda}{\partial y^\alpha} \frac{\partial x^\delta}{\partial y^\beta} \right]$

$= \det(g_{\alpha\beta}(x)) \operatorname{Tr}\left[\eta^{\mu\nu} \frac{\partial y^\kappa}{\partial x^\mu} \frac{\partial x^\lambda}{\partial x^\nu} \frac{\partial}{\partial x^\mu} \eta_{\lambda\delta} \frac{\partial x^\delta}{\partial y^\beta} \right]$

$= \det(g_{\alpha\beta}(x)) \operatorname{Tr}\left[\eta^\mu{}_\delta \frac{\partial y^\kappa}{\partial x^\mu} \frac{\partial}{\partial x^\mu} \frac{\partial x^\delta}{\partial y^\beta} \right]$

$= \det(g_{\alpha\beta}(x)) \operatorname{Tr}\left[\eta^\kappa{}_\beta \frac{\partial}{\partial x^\mu} \right]$

$= \det(g_{\alpha\beta}(x)) \cdot 4 \frac{\partial}{\partial x^\mu}$ as $\operatorname{Tr}[\mathbb{1}_4] = 4$

How to go on? Seems wrong to me to use H1 here — just one more step, see tutorial

d) Skew sym. tensorfield $w^{\nu_1 \dots \nu_k}(x)$

$$(\nabla w)^{\nu_1 \dots \nu_k}{}_{;\nu_1} = \frac{\partial}{\partial x^{\nu_1}} w^{\nu_1 \dots \nu_k} + w^{\alpha \nu_2 \dots \nu_k} \Gamma^{\nu_1}_{\alpha \nu_1} + \dots + w^{\nu_1 \nu_2 \dots \alpha} \Gamma^{\nu_k}_{\alpha \nu_1} \quad (*)$$

Also $\frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^{\nu_1}} \left(\sqrt{|g|}\, w^{\nu_1 \dots \nu_k} \right)$

$$= \frac{1}{\sqrt{|g|}} \left\{ \frac{1}{2\sqrt{|g|}} \left(\frac{\partial}{\partial x^{\nu_1}} |g| \right) w^{\nu_1 \dots \nu_k} + \sqrt{|g|} \frac{\partial}{\partial x^{\nu_1}} w^{\nu_1 \dots \nu_k} \right\}$$

$$= \frac{\partial}{\partial x^{\nu_1}} w^{\nu_1 \dots \nu_k} + \frac{1}{2|g|} \left(\frac{\partial}{\partial x^{\nu_1}} |g| \right) w^{\nu_1 \dots \nu_k}$$

$$\overset{c)}{=} \left(\frac{\partial}{\partial x^{\nu_1}} w^{\nu_1 \dots \nu_k} \right) + 2 w^{\nu_1 \dots \nu_k} \frac{\partial}{\partial x^{\nu_1}}$$

what is it acting on?

Now, taking a look at (*) again, we notice:

$$w^{\alpha \nu_2 \dots \nu_k} \Gamma^{\nu_1}_{\alpha \nu_1} = w^{\alpha \nu_2 \dots \nu_k} \frac{1}{2} g^{\nu_1 \delta} \left\{ \frac{\partial}{\partial x^\alpha} g_{\delta \nu_1} + \frac{\partial}{\partial x^{\nu_1}} g_{\delta \alpha} - \frac{\partial}{\partial x^\delta} g_{\alpha \nu_1} \right\}$$

$$= w^{\alpha \nu_2 \dots \nu_k} \frac{1}{2} \left\{ \underbrace{\frac{\partial}{\partial x^\alpha} \delta^{\nu_1}_{\nu_1}}_{=4 \frac{\partial}{\partial x^\alpha}} + \underbrace{\frac{\partial}{\partial x^{\nu_1}} \delta^{\nu_1}_{\alpha} - \frac{\partial}{\partial x^\delta} g_{\nu_1 \alpha} g^{\nu_1 \delta}}_{=0 \text{ w/ } w^{\alpha \nu_2 \dots \nu_k}} \right\}$$

Can not pull $g^{\nu_1 \delta}$ inside of derivative?

no

$$= 2 w^{\alpha \nu_2 \dots \nu_k} \frac{\partial}{\partial x^\alpha}$$

while e.g.

$$w^{\nu_1 \alpha \dots \nu_k} \Gamma^{\nu_2}_{\alpha \nu_1} = w^{\nu_1 \alpha \dots \nu_k} \frac{1}{2} g^{\nu_2 \delta} \left\{ \frac{\partial}{\partial x^\alpha} g_{\delta \nu_1} + \frac{\partial}{\partial x^{\nu_1}} g_{\delta \alpha} - \frac{\partial}{\partial x^\delta} g_{\nu_1 \alpha} \right\}$$

$$= w^{\nu_1 \alpha \dots \nu_k} \frac{1}{2} \left\{ \frac{\partial}{\partial x^\alpha} \delta^{\nu_2}_{\nu_1} + \frac{\partial}{\partial x^{\nu_1}} \delta^{\nu_2}_{\alpha} - \underbrace{\frac{\partial}{\partial x^\delta} g^{\nu_2 \delta} g_{\nu_1 \alpha}}_{\text{yields zero w/ } w^{\nu_1 \alpha \dots \nu_k} \text{ as it's skew sym.}} \right\}$$

1

$$= \frac{1}{2} \left\{ w^{\nu_2 \alpha \dots \nu_k} \frac{\partial}{\partial x^\alpha} + \overset{(-)}{w^{\nu_1 \nu_2 \dots \nu_k}} \frac{\partial}{\partial x^{\nu_1}} \right\} = 0$$

Thus $(\nabla w)^{\nu_1 \nu_2 \dots \nu_k}{}_{;\nu_1} = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^{\nu_1}} \left(\sqrt{|g|}\, w^{\nu_1 \dots \nu_k} \right)$

e) Putting $g_{\mu\nu}(\alpha) = g_{\mu\nu} + \alpha(\delta g)_{\mu\nu}$, we find

$$\frac{\partial}{\partial\alpha}|g(\alpha)|\Big|_{\alpha=0} = \det(g(\alpha))\,\mathrm{Tr}\left[g^{-1}(\alpha)\frac{\partial}{\partial\alpha}g(\alpha)\right]\Big|_{\alpha=0}$$

2

$$= \det(g(\alpha))\,\mathrm{Tr}\left[g^{-1}(\alpha)(\delta g)_{\mu\nu}\right]\Big|_{\alpha=0}$$

$$= \det(g(\alpha))\,\mathrm{Tr}\left[(g^{\kappa\mu}+O(\alpha))(\delta g)_{\mu\nu}\right]\Big|_{\alpha=0}$$

$$= \det(g(\alpha))\,g^{\kappa\mu}(\delta g)_{\mu\nu}$$

Not twice μ,ν as index? Trace! See tutorial

$$\mathrm{Tr}(A) = A^{\mu}{}_{\mu}$$

About the questions, you can ask them while we discuss the solutions or after the tutorial.