Disclaimer

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I raise no claim to correctness and completeness of the given solutions! This equally applies to the corrections mentioned above.

General Relativity 10. Exercise Marvin Zanke

25.06.2018 H13)

$$\frac{d}{d\tau} g_{\mu\nu} \dot{x}^\nu = \frac{1}{2} \frac{\partial}{\partial x^\mu} (g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta)$$

the geodesic equation

Had $\frac{d}{d\tau}(g_{\mu\nu}\dot{z}^\nu) = \frac{1}{2} g_{\alpha\beta,\mu} \dot{z}^\alpha \dot{z}^\beta$ ⇝ can put $\dot{z}^\alpha \dot{z}^\beta$ outside of $\partial/\partial x^\mu$ because no x-dep. - any ?? And $\frac{d}{d\tau}(g_{\mu\nu}\dot{x}^\nu)$ applies to both: $\frac{d}{d\tau}(g_{\mu\nu}\dot{x}^\nu)$? Yes. Yes.

a) Metric: $g_{00} = 1 + f(x^2, x^3)$, $g_{10} = g_{01} = -f(x^2, x^3)$

$g_{11} = -1 + f(x^2, x^3)$, $g_{22} = g_{33} = -1$

w/ $f(x^2, x^3) = (x^2)^2 - (x^3)^2$

[sorry for all cutting. But using is not a good idea Sol^n is not complete].

We are interested in the solutions $x^\mu(\tau)$ w/

- $g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta = 1 = \text{const}$
- $g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta = 0 = \text{const}$

~~something you impose after finding solution!~~ ~~OR Ignore this~~

⇝ geodesic equation: $\frac{d}{d\tau}(g_{\mu\nu}\dot{x}^\nu(\tau)) \cancel{=} 0 \cancel{=} g_{\mu\nu,\lambda}\dot{x}^\lambda \dot{x}^\nu + g_{\mu\nu}\ddot{x}^\nu$ How?

$\mu=0$: $g_{00,2}\dot{x}^0\dot{x}^2 + g_{00,3}\dot{x}^0\dot{x}^3 + g_{01,2}\dot{x}^1\dot{x}^2 + g_{01,3}\dot{x}^1\dot{x}^3$
$+ g_{00}\ddot{x}^0 + g_{01}\ddot{x}^1 = 0$

$\mu=1$: $g_{10,2}\dot{x}^0\dot{x}^2 + g_{10,3}\dot{x}^0\dot{x}^3 + g_{11,2}\dot{x}^1\dot{x}^2 + g_{11,3}\dot{x}^1\dot{x}^3$
$+ g_{10}\ddot{x}^0 + g_{11}\ddot{x}^1 = 0$

$\mu=2$: $g_{22}\ddot{x}^2 = 0$

$\mu=3$: $g_{33}\ddot{x}^3 = 0$

The last 2 equations yield: $\dot{x}^2 = c_2$ ⇝ $x^2 = c_2\tau + \tilde{c}_2$

and equiv. $x^3 = c_3\tau + \tilde{c}_3$, $\dot{x}^3 = c_3$

The first 2 eqs. give (coupled diff. eq.)

$$0 = 2x^2\dot{x}^0\dot{x}^2 - 2x^3\dot{x}^0\dot{x}^3 - 2x^2\dot{x}^1\dot{x}^2 + 2x^3\dot{x}^1\dot{x}^3$$
$$+ (1 + (x^2)^2 - (x^3)^2)\ddot{x}^0 - ((x^2)^2 - (x^3)^2)\ddot{x}^1$$

$$0 = -2x^2\dot{x}^0\dot{x}^2 + 2x^3\dot{x}^0\dot{x}^3 + 2x^2\dot{x}^1\dot{x}^2 - 2x^3\dot{x}^1\dot{x}^3$$
$$- ((x^2)^2 - (x^3)^2)\ddot{x}^0 + (-1 + (x^2)^2 - (x^3)^2)\ddot{x}^1$$

Adding those yields $0 = \ddot{x}^0 - \ddot{x}^1$

$$\rightsquigarrow \ddot{x}^0 = \ddot{x}^1 \quad \Leftrightarrow \quad \dot{x}^0 - \dot{x}^1 \equiv C_d$$

Inserting $0 = \ddot{x}^0 - \ddot{x}^1$ into the first equation yields

$$
\begin{aligned}
0 &= \ddot{x}^0 + 2x^2 \dot{x}^0 \dot{x}^2 - 2x^3 \dot{x}^0 \dot{x}^3 - 2x^2 \dot{x}^1 \dot{x}^2 + 2x^3 \dot{x}^1 \dot{x}^3 \\
&= \ddot{x}^0 + 2x^2 \dot{x}^2 (\dot{x}^0 - \dot{x}^1) - 2x^3 \dot{x}^3 (\dot{x}^0 - \dot{x}^1) \\
&= \ddot{x}^0 + C_d \{ 2x^2 \dot{x}^2 - 2x^3 \dot{x}^3 \} \\
&= \ddot{x}^0 + C_d \frac{d}{d\tau} \{ (x^2)^2 - (x^3)^2 \} = \ddot{x}^0 + C_d \frac{d}{d\tau} f(x^2, x^3)
\end{aligned}
$$

$$\rightsquigarrow \dot{x}^0 = -C_d f(x^2, x^3) + C_0$$

analogously, one finds $\dot{x}^1 = -C_d f(x^2, x^3) + C_1$
using the second equation

As $\dot{x}^0 - \dot{x}^1 = C_d$, we also have $C_d = C_0 - C_1$

Compare w/ sol^n in class.

We then look at $g_{\mu\nu}\dot{x}^\alpha\dot{x}^\beta = k$ for $k = 0, 1$

$$k = (1+f)\dot{x}^0\dot{x}^0 - 2f\dot{x}^0\dot{x}^1 + (f-1)\dot{x}^1\dot{x}^1 - \dot{x}^2\dot{x}^2 - \dot{x}^3\dot{x}^3$$

$$= (1+f)(C_0 - C_d f)^2 - 2f(C_0 - C_d f)(C_1 - C_d f) + (f-1)(C_1 - C_d f)^2 - C_2^2 - C_3^2$$

$$= (1+f)(C_0^2 + \cancel{C_d^2 f^2} - 2C_0 C_d f) - 2f(C_0 C_1 + \cancel{C_d^2 f^2} - C_0 C_d f - C_1 C_d f) + (f-1)(C_1^2 + \cancel{C_d^2 f^2} - 2C_1 C_d f) - C_2^2 - C_3^2$$

$$= C_0^2 - 2C_0 C_d f + C_0^2 f - \cancel{2C_0 C_d f^2} - 2f C_0 C_1 + \cancel{2C_0 C_d f^2} + \cancel{2C_1 C_d f^2} + C_1^2 f - \cancel{2C_1 C_d f^2} - C_1^2 + 2C_1 C_d f - C_2^2 - C_3^2$$

$$= C_0^2 - C_1^2 - C_2^2 - C_3^2 + C_0^2 f + C_1^2 f + 2f C_d (C_1 - C_0) - 2f C_0 C_1$$

$$= C_0^2(1+f) + C_1^2(f-1) - C_2^2 - C_3^2 + 2f C_d \underbrace{(C_1 - C_0)}_{-C_d} - 2f \underset{\uparrow\, C_d + C_1}{C_0} C_1$$

$$= C_0^2(1+f) - C_1^2(1+f) - C_2^2 - C_3^2 - 2f C_d (C_d + C_1)$$

$$= (1+f)\{C_0^2 - \underset{\uparrow\,(C_0 - C_d)^2}{C_1^2}\} - C_2^2 - C_3^2 - 2f C_d C_0$$

$$= (1+f)\{C_0^2 - (C_0^2 + C_d^2 - 2C_0 C_d)\} - C_2^2 - C_3^2 - 2f C_d C_0$$

$$= (1+f)\{-C_d^2 + 2C_0 C_d\} - C_2^2 - C_3^2 - 2f C_d C_0$$

$$= -(1+f) C_d^2 + 2\underset{\uparrow\, C_d + C_1}{C_0} C_d - C_2^2 - C_3^2$$

$$= (1-f) C_d^2 + 2C_1 C_d - C_2^2 - C_3^2$$

!?

Alternatively, we will be working in the coordinates

$$x^0(y^0, y^1) = \frac{y^0+y^1}{\sqrt{2}}, \quad x^1(y^0, y^1) = \frac{y^1-y^0}{\sqrt{2}}, \quad x^2(y^2) = x^2, \quad x^3(y^3) = x^3$$

from P. 11), in which the metric takes the form

$$\hat{g}_{\mu\nu}(y) = g_{\alpha\beta}(x(y)) \frac{\partial x^\alpha}{\partial y^\mu} \frac{\partial x^\beta}{\partial y^\nu}$$

$$\hat{g}_{00} = 2f(y), \quad \hat{g}_{01} = \hat{g}_{10} = 1, \quad \hat{g}_{22} = \hat{g}_{33} = -1$$

the inverse trafo is $y^0 = \frac{x^0 - x^1}{\sqrt{2}}$, $y^1 = \frac{x^0+x^1}{\sqrt{2}}$

$y^2 = x^2$, $y^3 = x^3$

$$\rightsquigarrow f(x^2, x^3) = (x^2)^2 - (x^3)^2 = (y^2)^2 - (y^3)^2 = f(y^2, y^3)$$

The conditions $g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta = 1$ and $g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta = 0$

become

$$\hat{g}_{\alpha\beta}\dot{y}^\alpha \dot{y}^\beta = 1 \text{ and } \hat{g}_{\alpha\beta}\dot{y}^\alpha \dot{y}^\beta = 0 \quad (*)$$

as the length

should be measured invariantly in both frames.

~~it is not a 4-vector.~~

The geodesic equation then becomes:

$$\frac{d}{dt}\hat{g}_{\mu\nu}\dot{y}^\nu = \frac{1}{2}\frac{\partial}{\partial x^\mu}(\hat{g}_{\alpha\beta}\dot{y}^\alpha \dot{y}^\beta) = 0$$

not a good idea.

$$\rightsquigarrow 0 = \frac{d}{dt}(\hat{g}_{\mu\nu}\dot{y}^\nu)$$

$\mu = 0$: $0 = \frac{d}{dt}\{2f(y)\dot{y}^0 + \dot{y}^1\} = \ddot{y}^1 + 2f(y)\ddot{y}^0 + 2\partial_\lambda f(y)\dot{y}^\lambda \dot{y}^0$

$\mu = 1$: $0 = \frac{d}{dt}\{\dot{y}^0\} = \ddot{y}^0$

$\mu = 2$: $0 = \frac{d}{dt}\{-1\dot{y}^2\} = -\ddot{y}^2$

$\mu = 3$: $0 = \frac{d}{dt}\{-1\dot{y}^3\} = -\ddot{y}^3$

One instantly finds using the last 3 eqs.

$$\overset{\circ\circ}{y}{}^0 = C_0 \rightsquigarrow y^0 = C_0\tau + \tilde{C}_0$$
$$\overset{\circ}{y}{}^2 = C_2 \rightsquigarrow y^2 = C_2\tau + \tilde{C}_2$$
$$\overset{\circ}{y}{}^3 = C_3 \rightsquigarrow y^3 = C_3\tau + \tilde{C}_3$$

+ more terms (sol^n not complete).

For y^1, one could use the first eq., i.e.

$$0 = \overset{\circ\circ}{y}{}^1 + 2f(y)\overset{\circ\circ}{y}{}^0_{=0} + 2\{\partial_2 f(y)\overset{\circ}{y}{}^2\overset{\circ}{y}{}^0 + \partial_3 f(y)\overset{\circ}{y}{}^3\overset{\circ}{y}{}^0\}$$
$$= \overset{\circ\circ}{y}{}^1 + 2\{2y^2\overset{\circ}{y}{}^2C_0 - 2y^3\overset{\circ}{y}{}^3C_0\} = \overset{\circ\circ}{y}{}^1 + 4C_0\{C_2^2\tau + C_2\tilde{C}_2 - C_3^2\tau - C_3\tilde{C}_3\}$$

but we want to find the properly normalized solutions using (*).

thus, $\hat{g}_{\alpha\beta}\overset{\circ}{y}{}^\alpha\overset{\circ}{y}{}^\beta = k$

$$\Leftrightarrow 0 = -k + 2f(y)\overset{\circ}{y}{}^0\overset{\circ}{y}{}^0 + 2\overset{\circ}{y}{}^0\overset{\circ}{y}{}^1 - \overset{\circ}{y}{}^2\overset{\circ}{y}{}^2 - \overset{\circ}{y}{}^3\overset{\circ}{y}{}^3$$
$$= -k + 2f(y)C_0^2 + 2C_0\overset{\circ}{y}{}^1 - C_2^2 - C_3^2$$
$$= -k + 2C_0^2\{(y^2)^2 - (y^3)^2\} + 2C_0\overset{\circ}{y}{}^1 - C_2^2 - C_3^2$$

$$\Leftrightarrow \overset{\circ}{y}{}^1 = \frac{1}{2C_0}\{k + C_2^2 + C_3^2 - 2C_0^2[(\overbrace{y^2}^{C_2\tau+\tilde{C}_2})^2 - (\overbrace{y^3}^{C_3\tau+\tilde{C}_3})^2]\}$$

$$\rightsquigarrow y^1 = \frac{1}{2C_0}\{k + C_2^2 + C_3^2\}\tau - C_0\{\frac{C_2^2}{3}\tau^3 + C_2\tilde{C}_2\tau^2 + \tilde{C}_2\tau - \frac{C_3^2}{3}\tau^3 - C_3\tilde{C}_3\tau^2 - \tilde{C}_3\tau\} + \tilde{C}_1$$

If I calculate $\overset{\circ\circ}{y}{}^1$ from this eq., I get $\overset{\circ\circ}{y}{}^1 = -2C_0\{y^2\overset{\circ}{y}{}^2 - y^3\overset{\circ}{y}{}^3\}$ which is not exactly equivalent to the eq. from ($\mu = 0$) above (factor 2 different)

↓ I didn't understand $\mu = 0$?

for $k = 0, 1$

Transforming again, one finds: $x^2 = C_2\tau + \tilde{C}_2$, $x^3 = C_3\tau + \tilde{C}_3$

$$x^0 = \frac{y^0 + y^1}{\sqrt{2}}, \quad x^1 = \frac{y^1 - y^0}{\sqrt{2}}$$

+ more terms

+ more terms