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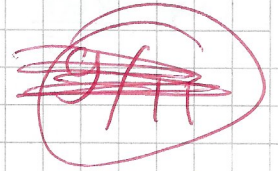
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# General Relativity Exercise 11

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02.07.2018 H14)  $x^0 = t, x^1 = r, x^2 = \vartheta, x^3 = \varphi$

$$g_{00} = 1, g_{11} = -\frac{R^2(t)}{1-kr^2}, g_{22} = -r^2 R^2(t), g_{33} = -r^2 R^2(t) \sin^2 \vartheta$$

$$\ddot{R}^2(t) + k = \frac{8\pi G}{3} \epsilon(t) R^2(t) \quad (*)$$

$$\frac{d}{dt} (\epsilon(t) R^3(t)) = -p(t) \frac{d}{dt} R^3(t) \quad (**)$$

Assume a universe dominated by radiation  $\rightarrow p = \frac{\epsilon}{3}$

(a): From (\*) we have  $\dot{\epsilon}(t) R^3(t) + 3R^2(t) \dot{R}(t) \epsilon(t) = -3p R^2(t) \dot{R}(t)$

$$\begin{aligned} \Rightarrow \dot{\epsilon}(t) &= -\epsilon(t) \frac{\dot{R}(t)}{R(t)} - 3 \frac{\dot{R}(t)}{R(t)} \epsilon(t) \\ &= -4 \frac{\dot{R}(t)}{R(t)} \epsilon(t) \end{aligned}$$

$$\Rightarrow \frac{\dot{\epsilon}(t)}{\epsilon(t)} = -4 \frac{\dot{R}(t)}{R(t)}$$

$$\Rightarrow \frac{d\epsilon}{\epsilon(t)} = -4 \frac{dR}{R(t)} \Rightarrow \int_{\epsilon(t_0)}^{\epsilon(t)} \frac{d\epsilon}{\epsilon(t)} = -4 \int_{R(t_0)}^{R(t)} \frac{dR}{R(t)}$$

$$\Rightarrow \ln \frac{\epsilon(t)}{\epsilon_0} = -4 \ln \frac{R(t)}{R_0}$$

$$\Rightarrow \frac{\epsilon(t)}{\epsilon_0} = \left( \frac{R(t)}{R_0} \right)^{-4}$$

2/2

(b): Rewriting (\*):  $H(t)^2 + \frac{k}{R(t)^2} = \frac{8\pi G}{3} \epsilon(t)$

Evaluating at  $t = t_0$ :

$$H_0^2 + \frac{k}{R_0^2} = \frac{8\pi G}{3} \epsilon_0 \Rightarrow k = R_0^2 \left\{ \frac{8\pi G}{3} \epsilon_0 - H_0^2 \right\}$$

tutorial:  
 $H_0^2 R_0^2 (2q_0 - 1)$

$$\Rightarrow \ddot{R}^2(t) = \frac{8\pi G}{3} \epsilon(t) R^2(t) - R_0^2 \left\{ \frac{8\pi G}{3} \epsilon_0 - H_0^2 \right\}$$

$$\begin{aligned} & \left| \epsilon(t) R^2(t) = \epsilon_0 \frac{R_0^4}{R^4(t)} \right. \\ & = \left. \frac{8\pi G}{3} \epsilon_0 \frac{R_0^4}{R^4(t)} - R_0^2 \left\{ \frac{8\pi G}{3} \epsilon_0 - H_0^2 \right\} \right. \end{aligned}$$

$$\Rightarrow \frac{\dot{R}(t)^2}{R_0^2} = \frac{8\pi G}{3} \epsilon_0 \frac{R_0^2}{R^2(t)} - \left\{ \frac{8\pi G}{3} \epsilon_0 - H_0^2 \right\}$$

$$\left| q_0 = \frac{4\pi}{3} \frac{\epsilon_0 \epsilon_0}{H_0^2} \right.$$

$$= 2q_0 H_0^2 \left\{ \frac{R_0^2}{R^2(t)} - 1 \right\} + H_0^2$$

$$= H_0^2 \left\{ 1 - 2q_0 + 2q_0 \frac{R_0^2}{R^2(t)} \right\}$$

2/2

(3): We then have

$$\left( \frac{dR}{dt} \right)^2 = H_0^2 \left\{ 1 - 2q_0 + 2q_0 \frac{R_0^2}{R^2(t)} \right\}$$

$$\Rightarrow (dt)^2 = \frac{(dR)^2}{R_0^2} \frac{1}{H_0^2} \left\{ 1 - 2q_0 + 2q_0 \frac{R_0^2}{R^2} \right\}^{-1} \quad (***)$$

$$\Rightarrow t = \frac{1}{R_0 H_0} \int_{R(t)}^{R_0} ds \left\{ 1 - 2q_0 + 2q_0 \frac{R_0^2}{s^2} \right\}^{-1/2}$$

$$\left| \frac{s^{-2}}{R_0^2} = s^{-2} \Rightarrow \frac{ds}{s} = \frac{1}{R_0} \right.$$

$$\left. = \frac{1}{H_0} \int ds \left\{ 1 - 2q_0 + 2q_0 \frac{1}{s^2} \right\}^{-1/2} \right. \quad \text{tag } \downarrow$$

(4): Starting from (\*\*\*) we find with (for  $q_0 > 1/2$ )

$$1 - \cos\theta = \frac{2q_0 - 1}{q_0} \frac{R^2}{R_0^2} \Rightarrow R^2 = R_0^2 (1 - \cos\theta) \frac{q_0}{2q_0 - 1}$$

$$\text{then} \quad \Rightarrow \frac{dR}{d\theta} = \frac{R_0}{2} \frac{\sin\theta}{1 - \cos\theta} \sqrt{\frac{q_0}{2q_0 - 1}}$$

$$dt = \frac{dR}{R_0 H_0} \left\{ 1 - 2q_0 + 2q_0 \frac{R_0^2}{R^2} \right\}^{-1/2}$$

$$= \frac{d\theta}{2H_0} \frac{\sin\theta}{1 - \cos\theta} \sqrt{\frac{q_0}{2q_0 - 1}} \left\{ 1 - 2q_0 + 2 \frac{2q_0 - 1}{1 - \cos\theta} \right\}^{-1/2}$$

$$= \frac{d\theta}{2H_0} \frac{\sin\theta}{1 - \cos\theta} q_0^{1/2} (2q_0 - 1)^{-1/2} (2q_0 - 1)^{-1/2} \left\{ -1 + \frac{2}{1 - \cos\theta} \right\}^{-1/2}$$

$$= \frac{d\theta}{2H_0} \sin\theta q_0^{1/2} (2q_0 - 1)^{-1} \left\{ \cos\theta - 1 + 2 \right\}^{-1/2}$$

Okay to "max" (University) w/ the differential like this?

Where from  $\epsilon_0$  for  $q_0$ ?  
 $\Rightarrow$  can only be solved for the different cases separately

Didn't use  $q_0 > 1/2$  here?

and thus

$$t = \frac{1}{2H_0} q_0^{1/2} (2q_0 - 1)^{-1} \int d\theta \frac{\sin \theta}{\sqrt{1 + \cos \theta}}$$

$$\left| \int d\theta \frac{\sin \theta}{\sqrt{1 + \cos \theta}} \stackrel{z = 1 + \cos \theta}{\frac{dz}{d\theta} = -\sin \theta} - \int dz z^{-1/2} = -2z^{1/2} \right.$$

$$= \frac{1}{2H_0} q_0^{1/2} (2q_0 - 1)^{-1} \left. - 2\sqrt{1 + \cos \theta} \right|_0^{\theta}$$

$$= \frac{1}{H_0} q_0^{1/2} (2q_0 - 1)^{-1} \left[ \sqrt{2} - \sqrt{1 + \cos \theta} \right]$$

Why boundaries for  $\theta$  integral like this?

(1)

(e) Consider the case  $q_0 = 1/2$ .

insert substitution back in to get stl. like in tutorial (+ inverse)

From (c):  $t = \frac{1}{H_0} \int_0^{R(t)/R_0} dy \left[ 1 - 2q_0 + 2q_0 \frac{1}{y^2} \right]^{-1/2}$

$$= \frac{1}{H_0} \int_0^{R(t)/R_0} dy \left[ \frac{1}{y^2} \right]^{-1/2} = \frac{1}{H_0} \int_0^{R(t)/R_0} dy y$$

$$= \frac{1}{2H_0} \left( \frac{R(t)}{R_0} \right)^2$$

$\Rightarrow \frac{R(t)}{R_0} = (2H_0 t)^{1/2}$

(i)

(f) For the case  $0 < q_0 < 1/2$ , we define

$$1 - \cos \psi = \frac{2q_0 - 1}{q_0} \frac{R^2}{R_0^2} \Rightarrow R^2 = R_0^2 (1 - \cos \psi) \frac{q_0}{2q_0 - 1}$$

$$\Rightarrow \frac{dR}{d\psi} = - \frac{R_0}{2} \frac{\sin \psi}{\sqrt{1 - \cos \psi}} \sqrt{\frac{q_0}{2q_0 - 1}}$$

and from (a), we find

Factor  $(2q_0 - 1)^{-1/2}$  out here? But it's  $< 0$  as  $q_0 \in (0, 1/2)^2$

$$dt = \frac{dR}{R_0 H_0} \left[ 1 - 2q_0 + 2q_0 \frac{R_0^2}{R^2} \right]^{-1/2}$$

$$= - \frac{d\psi}{2H_0} \frac{\sin \psi}{\sqrt{1 - \cos \psi}} q_0^{1/2} (2q_0 - 1)^{-1/2} \left[ 1 - 2q_0 + 2 \frac{2q_0 - 1}{1 - \cos \psi} \right]^{-1/2}$$

$$= - \frac{d\psi}{2H_0} q_0^{1/2} (2q_0 - 1)^{-1} \frac{\sin \psi}{\sqrt{1 - \cos \psi}} \left[ -1 + \frac{2}{1 - \cos \psi} \right]^{-1/2}$$

$$= + \frac{d\psi}{2H_0} q_0^{1/2} (1 - 2q_0)^{-1} \sin \psi \left[ \cos \psi - 1 + 2 \right]^{-1/2}$$

Ans, one has

$$t = \frac{1}{2H_0} q_0^{1/2} (1-2q_0)^{-1} \int d\phi \frac{\sinh\phi}{\sqrt{1+\cosh\phi}}$$

$$\left| \int d\phi \frac{\sinh\phi}{\sqrt{1+\cosh\phi}} \quad \begin{array}{l} z = 1 + \cosh\phi \\ dz = \sinh\phi \end{array} \right. \int dz z^{-1/2} = 2z^{1/2}$$

$$= \frac{1}{H_0} q_0^{1/2} (1-2q_0)^{-1} \left. \sqrt{1+\cosh\phi} \right|_0^\phi$$

$$= \frac{1}{H_0} q_0^{1/2} (1-2q_0)^{-1} \left\{ \sqrt{1+\cosh\phi} - \sqrt{2} \right\}$$

$$\Rightarrow H_0 t = q_0^{1/2} (1-2q_0)^{-1} \left\{ \sqrt{1+\cosh\phi} - \sqrt{2} \right\}$$

