Disclaimer

General Relativity Exercise 11

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02.07.2018 H14) $x^0 = t,\ x^1 = r,\ x^2 = \vartheta,\ x^3 = \varphi$

$$g_{00} = 1,\ g_{11} = -\frac{R^2(t)}{1-kr^2},\ g_{22} = -r^2R^2(t),\ g_{33} = -r^2R^2(t)\sin^2\vartheta$$

$$\dot{R}^2(t) + k = \frac{8\pi G}{3}\epsilon(t)R^2(t) \quad (*)$$

$$\frac{d}{dt}\left(\epsilon(t)R^3(t)\right) = -p(t)\frac{d}{dt}R^3(t) \quad (**)$$

Assume a universe dominated by radiation $\rightarrow p = \frac{\epsilon}{3}$

("a"): From $(**)$ $\rightsquigarrow$ $\dot{\epsilon}(t)R^3(t) + 3R^2(t)\dot{R}(t)\epsilon(t) = -3pR^2(t)\dot{R}(t)$

$$\rightsquigarrow \dot{\epsilon}(t) = -\epsilon(t)\frac{\dot{R}(t)}{R(t)} - 3\frac{\dot{R}(t)}{R(t)}\epsilon(t)$$

$$= -4\frac{\dot{R}(t)}{R(t)}\epsilon(t)$$

$$\rightsquigarrow \frac{\dot{\epsilon}(t)}{\epsilon(t)} = -4\frac{\dot{R}(t)}{R(t)},$$

$$\rightsquigarrow \frac{d\epsilon}{\epsilon(t)} = -4\frac{dR}{R(t)} \rightsquigarrow \int_{\epsilon(t_0)}^{\epsilon(t)}\frac{d\epsilon}{\epsilon(t)} = -4\int_{R(t_0)}^{R(t)}\frac{dR}{R(t)}$$

$$\rightsquigarrow \ln\frac{\epsilon(t)}{\epsilon_0} = -4\ln\frac{R(t)}{R_0}$$

$$\rightsquigarrow \frac{\epsilon(t)}{\epsilon_0} = \left(\frac{R(t)}{R_0}\right)^{-4}$$

2/2

("b"): Rewriting $(*)$: $H(t)^2 + \frac{k}{R(t)^2} = \frac{8\pi G}{3}\epsilon(t)$

evaluating at $t = t_0$:

tutorial: $/\!/\ H_0^2R_0^2(2q_0 - 1)$

$$H_0^2 + \frac{k}{R_0^2} = \frac{8\pi G}{3}\epsilon_0 \rightsquigarrow k = R_0^2\left\{\frac{8\pi G}{3}\epsilon_0 - H_0^2\right\}$$

$$\overset{(*)}{\rightsquigarrow} \dot{R}^2(t) = \frac{8\pi G}{3}\epsilon(t)R^2(t) - R_0^2\left\{\frac{8\pi G}{3}\epsilon_0 - H_0^2\right\}$$

$$\left|\ \epsilon(t)R^2(t) = \epsilon_0\frac{R_0^4}{R^2(t)}\right.$$

$$= \frac{8\pi G}{3}\epsilon_0\frac{R_0^4}{R^2(t)} - R_0^2\left\{\frac{8\pi G}{3}\epsilon_0 - H_0^2\right\}$$

$$\rightsquigarrow \frac{\dot{R}(t)^2}{R_0^2} = \frac{8\pi G}{3} \epsilon_0 \frac{R_0^2}{R^2(t)} - \left\{ \frac{8\pi G}{3} \epsilon_0 - H_0^2 \right\}$$

$$\Big| \; q_0 = \frac{4\pi}{3} \frac{G \epsilon_0}{H_0^2}$$

$$= 2 q_0 H_0^2 \left\{ \frac{R_0^2}{R^2(t)} - 1 \right\} + H_0^2$$

$$= H_0^2 \left\{ 1 - 2q_0 + 2q_0 \frac{R_0^2}{R^2(t)} \right\}$$

2½

"c": We then have

$$\frac{\left(\frac{dR}{dt}\right)^2}{R_0^2} = H_0^2 \left\{ 1 - 2q_0 + 2q_0 \frac{R_0^2}{R(t)^2} \right\}$$

$$\rightsquigarrow (dt)^2 = \frac{(dR)^2}{R_0^2} \frac{1}{H_0^2} \left\{ 1 - 2q_0 + 2q_0 \frac{R_0^2}{R(t)^2} \right\}^{-1} \quad (***)$$

Okay to "mess" (unrigorously) w/ the differentials like this?

$$\rightsquigarrow t = \frac{1}{R_0 H_0} \int_{R(0)}^{R(t)} d\xi \left\{ 1 - 2q_0 + 2q_0 \frac{R_0^2}{\xi^2} \right\}^{-1/2}$$

$$\Big| \; \frac{\xi^{2}}{R_0^2} = \rho^2 \rightsquigarrow \frac{d\rho}{d\xi} = \frac{1}{R_0}$$

2½

$$= \frac{1}{H_0} \int_0^{R(t)/R_0} d\rho \left\{ 1 - 2q_0 + 2q_0 \frac{1}{\rho^2} \right\}^{-1/2} \rightsquigarrow t_{age} \overset{R(t)=R_0}{=}$$

Where from ϵ_0 for q_0? ⇝ can only be solved for the different cases separately

"d": Starting from (***), we find with (for $q_0 > 1/2$)

$$1 - \cos\theta = \frac{2q_0 - 1}{q_0} \frac{R^2}{R_0^2} \rightsquigarrow R^2 = R_0^2 (1 - \cos\theta) \frac{q_0}{2q_0 - 1}$$

then

$$\rightsquigarrow \frac{dR}{d\theta} = \frac{R_0}{2} \frac{\sin\theta}{\sqrt{1 - \cos\theta}} \sqrt{\frac{q_0}{2q_0 - 1}}$$

$$dt = \frac{dR}{R_0 H_0} \left\{ 1 - 2q_0 + 2q_0 \frac{R_0^2}{R(t)^2} \right\}^{-1/2}$$

$$= \frac{d\theta}{2H_0} \frac{\sin\theta}{\sqrt{1 - \cos\theta}} \sqrt{\frac{q_0}{2q_0 - 1}} \left\{ 1 - 2q_0 + 2 \frac{2q_0 - 1}{1 - \cos\theta} \right\}^{-1/2}$$

$$= \frac{d\theta}{2H_0} \frac{\sin\theta}{\sqrt{1 - \cos\theta}} q_0^{1/2} (2q_0 - 1)^{-1/2} (2q_0 - 1)^{-1/2} \left\{ -1 + \frac{2}{1 - \cos\theta} \right\}^{-1/2}$$

Didn't use $q_0 > 1/2$ here?

$$= \frac{d\theta}{2H_0} \sin\theta \, q_0^{1/2} (2q_0 - 1)^{-1} \left\{ \cos\theta - 1 + 2 \right\}^{-1/2}$$

and thus

$$t = \frac{1}{2H_0} q_0^{1/2} (2q_0-1)^{-1} \int d\theta \, \frac{\sin\theta}{\sqrt{1+\cos\theta}}$$

$$\int d\theta \frac{\sin\theta}{\sqrt{1+\cos\theta}} \overset{z=1+\cos\theta}{\underset{\frac{dz}{d\theta}=-\sin\theta}{=}} -\int dz\, z^{-1/2} = -2z^{1/2}$$

why boundaries for θ integral like this?

$$= \frac{1}{2H_0} q_0^{1/2}(2q_0-1)^{-1} \left\{-2\sqrt{1+\cos\theta}\right\}_0^{\theta}$$

$$= \frac{1}{H_0} q_0^{1/2}(2q_0-1)^{-1}\left\{\sqrt{2} - \sqrt{1+\cos\theta}\right\}$$

(+)

insert substitution back in to get sle. like in tutorial (+ inverse)

("e") Consider the case $q_0 = 1/2$:

From ("c"): $t = \frac{1}{H_0}\int_0^{R(t)/R_0} d\rho \left\{1-2q_0+2q_0\,\rho^2\right\}^{-1/2}$

$$= \frac{1}{H_0}\int_0^{R(t)/R_0} d\rho \left\{\frac{1}{\rho^2}\right\}^{-1/2} = \frac{1}{H_0}\int_0^{R(t)/R_0} d\rho\, \rho$$

$$= \frac{1}{2H_0}\left(\frac{R(t)}{R_0}\right)^2$$

$$\rightsquigarrow \frac{R(t)}{R_0} = (2H_0 t)^{1/2}$$

("f"): For the case $0<q_0<1/2$, we define

$$1-\cosh\phi = \frac{2q_0-1}{q_0}\frac{R^2}{R_0^2} \rightsquigarrow R^2 = R_0^2(1-\cosh\phi)\frac{q_0}{2q_0-1}$$

$$\rightsquigarrow \frac{dR}{d\phi} = -\frac{R_0}{2}\frac{\sinh\phi}{\sqrt{1-\cosh\phi}}\sqrt{\frac{q_0}{2q_0-1}}$$

and from (***), we find

Factor $(2q_0-1)^{-1/2}$ out here? But it's <0 as $q_0\in(0,1/2)$?

$$dt = \frac{dR}{R_0 H_0}\left\{1-2q_0+2q_0\frac{R_0^2}{R(t)^2}\right\}^{-1/2}$$

$$= -\frac{d\phi}{2H_0}\frac{\sinh\phi}{\sqrt{1-\cosh\phi}}\, q_0^{1/2}(2q_0-1)^{-1/2}\left\{1-2q_0+2\frac{2q_0-1}{1-\cosh\phi}\right\}^{-1/2}$$

$$= -\frac{d\phi}{2H_0} q_0^{1/2}(2q_0-1)^{-1}\frac{\sinh\phi}{\sqrt{1-\cosh\phi}}\left\{-1+\frac{2}{1-\cosh\phi}\right\}^{-1/2}$$

$$= +\frac{d\phi}{2H_0} q_0^{1/2}(1-2q_0)^{-1}\sinh\phi\left\{\cosh\phi-1+2\right\}^{-1/2}$$

Thus, one has

$$t = \frac{1}{2H_0} q_0^{1/2} (1-2q_0)^{-1} \int d\phi \, \frac{\sinh\phi}{\sqrt{1+\cosh\phi}}$$

$$\int d\phi \, \frac{\sinh\phi}{\sqrt{1+\cosh\phi}} \underset{\frac{dz}{d\phi} = \sinh\phi}{\overset{z = 1+\cosh\phi}{=}} \int dz \, z^{-1/2} = 2 z^{1/2}$$

$$= \frac{1}{H_0} q_0^{1/2} (1-2q_0)^{-1} \left\{ \sqrt{1+\cosh\phi} \right\}_0^{\phi}$$

$$= \frac{1}{H_0} q_0^{1/2} (1-2q_0)^{-1} \left\{ \sqrt{1+\cosh\phi} - \sqrt{2} \right\}$$

$$\rightsquigarrow H_0 t = q_0^{1/2} (1-2q_0)^{-1} \left\{ \sqrt{1+\cosh\phi} - \sqrt{2} \right\}$$