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General Relativity Exercise 2 Homework   Marvin Zanke

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H4) We have $\Gamma^\alpha_{\beta\gamma}(g(x)) \overset{\text{suppressed here}}{=} \frac{1}{2} g^{\alpha\delta}\left(\frac{\partial}{\partial x^\beta} g_{\delta\gamma} + \frac{\partial}{\partial x^\gamma} g_{\delta\beta} - \frac{\partial}{\partial x^\delta} g_{\beta\gamma}\right)$

a) $$\Gamma^\alpha_{\beta\gamma}(\varphi^* g(x)) = \frac{1}{2}(\varphi^* g)^{\alpha\delta}\left(\frac{\partial}{\partial x^\beta}(\varphi^* g)_{\delta\gamma} + \frac{\partial}{\partial x^\gamma}(\varphi^* g)_{\delta\beta} - \frac{\partial}{\partial x^\delta}(\varphi^* g)_{\beta\gamma}\right)$$

$$\overset{y=\varphi(x)}{=} \frac{1}{2} g^{k_1\epsilon_1}(y) \frac{\partial x^\alpha}{\partial y^{k_1}} \frac{\partial x^\delta}{\partial y^{\epsilon_1}} \Big\{ \frac{\partial}{\partial x^\beta}\left[g_{k_2\epsilon_2}(y) \frac{\partial y^{k_2}}{\partial x^\delta} \frac{\partial y^{\epsilon_2}}{\partial x^\gamma}\right]$$
$$+ \frac{\partial}{\partial x^\gamma}\left[g_{k_3\epsilon_3}(y) \frac{\partial y^{k_3}}{\partial x^\delta} \frac{\partial y^{\epsilon_3}}{\partial x^\beta}\right] - \frac{\partial}{\partial x^\delta}\left[g_{k_4\epsilon_4}(y) \frac{\partial y^{k_4}}{\partial x^\beta} \frac{\partial y^{\epsilon_4}}{\partial x^\gamma}\right]\Big\}$$

$$= \frac{1}{2} g^{k_1\epsilon_1}(y) \frac{\partial x^\alpha}{\partial y^{k_1}} \frac{\partial x^\delta}{\partial y^{\epsilon_1}} \Big\{ \frac{\partial y^{k_2}}{\partial x^\delta} \frac{\partial y^{\epsilon_2}}{\partial x^\gamma} \frac{\partial y^\lambda}{\partial x^\beta} \frac{\partial}{\partial y^\lambda} g_{k_2\epsilon_2}(y)$$
$$+ \frac{\partial y^{k_3}}{\partial x^\delta} \frac{\partial y^{\epsilon_3}}{\partial x^\beta} \frac{\partial y^\lambda}{\partial x^\gamma} \frac{\partial}{\partial y^\lambda} g_{k_3\epsilon_3}(y) - \frac{\partial y^{k_4}}{\partial x^\beta} \frac{\partial y^{\epsilon_4}}{\partial x^\gamma} \frac{\partial y^\lambda}{\partial x^\delta} \frac{\partial}{\partial y^\lambda} g_{k_4\epsilon_4}(y)$$
$$+ g_{k_2\epsilon_2}(y) \frac{\partial^2 y^{k_2}}{\partial x^\beta \partial x^\delta} \frac{\partial y^{\epsilon_2}}{\partial x^\gamma} + g_{k_2\epsilon_2}(y) \frac{\partial^2 y^{\epsilon_2}}{\partial x^\beta \partial x^\gamma} \frac{\partial y^{k_2}}{\partial x^\delta}$$
$$+ g_{k_3\epsilon_3}(y) \frac{\partial^2 y^{k_3}}{\partial x^\gamma \partial x^\delta} \frac{\partial y^{\epsilon_3}}{\partial x^\beta} + g_{k_3\epsilon_3}(y) \frac{\partial^2 y^{\epsilon_3}}{\partial x^\gamma \partial x^\beta} \frac{\partial y^{k_3}}{\partial x^\delta}$$
$$- g_{k_4\epsilon_4}(y) \frac{\partial^2 y^{k_4}}{\partial x^\delta \partial x^\beta} \frac{\partial y^{\epsilon_4}}{\partial x^\gamma} - g_{k_4\epsilon_4}(y) \frac{\partial^2 y^{\epsilon_4}}{\partial x^\delta \partial x^\gamma} \frac{\partial y^{k_4}}{\partial x^\beta}\Big\}$$

In which cases can we also "cancel" sth. like $\frac{\partial x^\delta}{\partial x^k}$? only iff $\delta = k$?

$\frac{\partial x^\delta}{\partial y^\epsilon} \frac{\partial y^k}{\partial x^\delta} = \delta^k_\epsilon$

$$\boxed{\frac{\partial x^\delta}{\partial x^k} = \delta^\delta_k}$$

but $\frac{\partial x^\epsilon}{\partial y^k} \frac{\partial y^\rho}{\partial x^\delta} + \frac{\partial x^\epsilon}{\partial x^\delta} \frac{\partial y^\rho}{\partial y^k}$ not possible only if $\epsilon = \delta$, then it's kind of the chain rule.

$$= \frac{1}{2} g^{k_1\epsilon_1}(y) \Big\{ \frac{\partial x^\alpha}{\partial y^{k_1}} \delta^{k_2}_{\epsilon_1} \frac{\partial y^{\epsilon_2}}{\partial x^\gamma} \frac{\partial y^\lambda}{\partial x^\beta} \frac{\partial}{\partial y^\lambda} g_{k_2\epsilon_2}(y)$$
$$+ \frac{\partial x^\alpha}{\partial y^{k_1}} \delta^{k_3}_{\epsilon_1} \frac{\partial y^{\epsilon_3}}{\partial x^\beta} \frac{\partial y^\lambda}{\partial x^\gamma} \frac{\partial}{\partial y^\lambda} g_{k_3\epsilon_3}(y) - \frac{\partial x^\alpha}{\partial y^{k_1}} \delta^\lambda_{\epsilon_1} \frac{\partial y^{k_4}}{\partial x^\beta} \frac{\partial y^{\epsilon_4}}{\partial x^\gamma} \frac{\partial}{\partial y^\lambda} g_{k_4\epsilon_4}(y)$$
$$+ 2 g_{k\epsilon}(y) \frac{\partial^2 y^\epsilon}{\partial x^\beta \partial x^\gamma} \frac{\partial y^k}{\partial x^\delta} \frac{\partial x^\alpha}{\partial y^{k_1}} \frac{\partial x^\delta}{\partial y^{\epsilon_1}}\Big\}$$

$$\overset{\substack{\lambda \leftrightarrow \epsilon_3 = \epsilon \\ \text{rename and } \epsilon_2 = \epsilon}}{=} \frac{1}{2} g^{k_1\epsilon_1}(y) \Big\{ \frac{\partial x^\alpha}{\partial y^{k_1}} \frac{\partial y^\epsilon}{\partial x^\gamma} \frac{\partial y^\lambda}{\partial x^\beta} \frac{\partial}{\partial y^\lambda} g_{\epsilon_1\epsilon}(y)$$
$$+ \frac{\partial x^\alpha}{\partial y^{k_1}} \frac{\partial y^\lambda}{\partial x^\beta} \frac{\partial y^\epsilon}{\partial x^\gamma} \frac{\partial}{\partial y^\epsilon} g_{\epsilon_1\lambda}(y) - \frac{\partial x^\alpha}{\partial y^{k_1}} \frac{\partial y^{k_4}}{\partial x^\beta} \frac{\partial y^{\epsilon_4}}{\partial x^\gamma} \frac{\partial}{\partial y^{\epsilon_1}} g_{k_4\epsilon_4}(y)\Big\}$$
$$+ g^{k_1\epsilon_1}(y) g_{k\epsilon}(y) \frac{\partial^2 y^\epsilon}{\partial x^\beta \partial x^\gamma} \delta^k_{\epsilon_1} \frac{\partial x^\alpha}{\partial y^{k_1}}$$

$$\overset{\substack{k_4 = \lambda \\ \epsilon_4 = \epsilon}}{=} \frac{1}{2} g^{k_1\epsilon_1}(y) \Big\{ \frac{\partial}{\partial y^\lambda} g_{\epsilon_1\epsilon}(y) + \frac{\partial}{\partial y^\epsilon} g_{\epsilon_1\lambda} - \frac{\partial}{\partial y^{\epsilon_1}} g_{\lambda\epsilon}(y)\Big\} \frac{\partial x^\alpha}{\partial y^{k_1}} \frac{\partial y^\lambda}{\partial x^\beta} \frac{\partial y^\epsilon}{\partial x^\gamma}$$
$$+ g^{k_1\epsilon_1}(y) g_{\epsilon_1\epsilon}(y) \frac{\partial^2 y^\epsilon}{\partial x^\beta \partial x^\gamma} \frac{\partial x^\alpha}{\partial y^{k_1}}$$

$$\overset{k_1 = k}{=} \Gamma^k_{\lambda\epsilon}(g(y)) \frac{\partial x^\alpha}{\partial y^k} \frac{\partial y^\lambda}{\partial x^\beta} \frac{\partial y^\epsilon}{\partial x^\gamma} + \frac{\partial x^\alpha}{\partial y^k} \delta^k_\epsilon \frac{\partial^2 y^\epsilon}{\partial x^\beta \partial x^\gamma}$$

$$= \Gamma^k_{\lambda\epsilon}(g(y)) \frac{\partial x^\alpha}{\partial y^k} \frac{\partial y^\lambda}{\partial x^\beta} \frac{\partial y^\epsilon}{\partial x^\gamma} + \frac{\partial x^\alpha}{\partial y^\epsilon} \frac{\partial^2 y^\epsilon}{\partial x^\beta \partial x^\gamma}$$

b) For an object to transform like tensorfields, we would expect

$$(\varphi^*\Gamma)^{\alpha}_{\beta\gamma}(g(x)) = \Gamma^{\kappa}_{\lambda\epsilon}(g(y)) \frac{\partial y^{\lambda}}{\partial x^{\beta}} \frac{\partial y^{\epsilon}}{\partial x^{\gamma}} \frac{\partial x^{\alpha}}{\partial y^{\kappa}}$$

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which is not the case because of the additional term

$$\frac{\partial x^{\alpha}}{\partial y^{\epsilon}} \frac{\partial^2 y^{\epsilon}}{\partial x^{\beta}\partial x^{\gamma}}$$

as seen from a)

why do we check $(\varphi^*\Gamma)(g(x)) \neq \Gamma(\varphi^* g(x))$ for a transform like a tensor field? And why then not apply φ^* to g as well in $\varphi^*\Gamma$, as g is in Γ? Could basically both be true but two different things?

we check

$$(\varphi^*\Gamma)^{\alpha}_{\beta\gamma} = \Gamma^{\kappa}_{\lambda\epsilon}(\varphi(x)) \frac{\partial y^{\lambda}}{\partial x^{\beta}} \frac{\partial y^{\epsilon}}{\partial x^{\gamma}} \frac{\partial x^{\alpha}}{\partial y^{\kappa}}$$

~~not~~ but ~~$(\varphi^*\Gamma)(g(x)) \neq \Gamma(\varphi^* g(x))$~~

→ So, here we don't know beforehand if Γ is a tensor, so we use the fact that g is a tensor to transform g first and then apply Γ i.e. we find $\Gamma(\varphi^* g)$.

∴ we cannot have an operation $\varphi^*\Gamma$ defined here.

H5) Take a contravariant tensorfield of rank 1, X^ρ

① a) Claim: $X^\rho{}_{;\nu;\mu} - X^\rho{}_{;\mu;\nu} = R^\rho{}_{\alpha\mu\nu} X^\alpha$

X^ρ vectorfield ? depends on x as well?

Yes!

Calculate

$$X^\rho{}_{;\nu} = \frac{\partial}{\partial x^\nu} X^\rho + X^\alpha \Gamma^\rho_{\alpha\nu}, \quad X^\rho{}_{;\mu} = \frac{\partial}{\partial x^\mu} X^\rho + X^\alpha \Gamma^\rho_{\alpha\mu}$$

$$X^\rho{}_{;\nu;\mu} = \frac{\partial}{\partial x^\mu} X^\rho{}_{;\nu} + X^\alpha{}_{;\nu} \Gamma^\rho_{\alpha\mu} - X^\rho{}_{;\alpha} \Gamma^\alpha_{\nu\mu}$$

$$\rightsquigarrow X^\rho{}_{;\mu;\nu} = \frac{\partial}{\partial x^\nu} X^\rho{}_{;\mu} + X^\alpha{}_{;\mu} \Gamma^\rho_{\alpha\nu} - X^\rho{}_{;\alpha} \Gamma^\alpha_{\mu\nu}$$

Take care ∇! Don't know how ∇_μ acts on $\partial_\nu X^\rho$ → order is important!

$$\Rightarrow X^\rho{}_{;\nu;\mu} - X^\rho{}_{;\mu;\nu} = \frac{\partial}{\partial x^\mu} X^\rho{}_{;\nu} - \frac{\partial}{\partial x^\nu} X^\rho{}_{;\mu} + X^\alpha{}_{;\nu}\Gamma^\rho_{\alpha\mu} - X^\alpha{}_{;\mu}\Gamma^\rho_{\alpha\nu} - X^\rho{}_{;\alpha}\Gamma^\alpha_{\nu\mu} + X^\rho{}_{;\alpha}\Gamma^\alpha_{\mu\nu}$$

$$\overset{\Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\nu\mu}}{=} \frac{\partial}{\partial x^\mu} X^\rho{}_{;\nu} - \frac{\partial}{\partial x^\nu} X^\rho{}_{;\mu} + X^\alpha{}_{;\nu}\Gamma^\rho_{\alpha\mu} - X^\alpha{}_{;\mu}\Gamma^\rho_{\alpha\nu}$$

?

Why is R the curvature tensor? Pictorially $X^\rho{}_{;\nu;\mu} - X^\rho{}_{;\mu;\nu}$?

→ in tut.

$$= \frac{\partial}{\partial x^\mu}\left\{\frac{\partial}{\partial x^\nu} X^\rho + X^\alpha \Gamma^\rho_{\alpha\nu}\right\} - \frac{\partial}{\partial x^\nu}\left\{\frac{\partial}{\partial x^\mu} X^\rho + X^\alpha\Gamma^\rho_{\alpha\mu}\right\} + \left\{\frac{\partial}{\partial x^\nu} X^\alpha + X^\kappa \Gamma^\alpha_{\kappa\nu}\right\}\Gamma^\rho_{\alpha\mu} - \left\{\frac{\partial}{\partial x^\mu} X^\alpha + X^\kappa\Gamma^\alpha_{\kappa\mu}\right\}\Gamma^\rho_{\alpha\nu}$$

$$= \frac{\partial}{\partial x^\mu}\left(X^\alpha \Gamma^\rho_{\alpha\nu}\right) - \frac{\partial}{\partial x^\nu}\left(X^\alpha\Gamma^\rho_{\alpha\mu}\right) + \left(\frac{\partial}{\partial x^\nu}X^\alpha\right)\Gamma^\rho_{\alpha\mu} - \left(\frac{\partial}{\partial x^\mu}X^\alpha\right)\Gamma^\rho_{\alpha\nu} + X^\kappa\Gamma^\alpha_{\kappa\nu}\Gamma^\rho_{\alpha\mu} - X^\kappa\Gamma^\alpha_{\kappa\mu}\Gamma^\rho_{\alpha\nu}$$

$$= X^\alpha \frac{\partial}{\partial x^\mu}\Gamma^\rho_{\alpha\nu} - X^\alpha\frac{\partial}{\partial x^\nu}\Gamma^\rho_{\alpha\mu} + X^\alpha\Gamma^\delta_{\alpha\nu}\Gamma^\rho_{\delta\mu} - X^\alpha\Gamma^\delta_{\alpha\mu}\Gamma^\rho_{\delta\nu}$$

?

Which order of the indices on the curvature tensor? Could also be $R^\rho{}_{\mu\alpha\nu}$ but ≠ $R^\rho{}_{\alpha\mu\nu}$?

$$= \left\{\frac{\partial}{\partial x^\mu}\Gamma^\rho_{\alpha\nu} - \frac{\partial}{\partial x^\nu}\Gamma^\rho_{\alpha\mu} + \Gamma^\delta_{\alpha\nu}\Gamma^\rho_{\delta\mu} - \Gamma^\varepsilon_{\alpha\mu}\Gamma^\rho_{\varepsilon\nu}\right\} X^\alpha$$

$$\equiv R^\rho{}_{\alpha\mu\nu} X^\alpha$$

just a definition

② b) As we found $R^\rho{}_{\alpha\mu\nu}$ via $X^\rho{}_{;\nu;\mu} - X^\rho{}_{;\mu;\nu}$ and $X^\rho{}_{;\nu;\mu} = (\nabla\nabla X)^\rho_{\nu\mu}$, $X^\rho{}_{;\mu;\nu} = (\nabla\nabla X)^\rho_{\mu\nu}$, and we know that for a tensorfield $w(x) \in V^{(p,q)}$ the covariant derivative yields a tensorfield $(\nabla w)(x) \in V^{(p+1,q)}$, we know that for $X^\rho \in V^{(0,1)}$ $(\nabla\nabla X)^\rho_{\mu\nu} \in V^{(2,1)}$. We also know that for the product of $w(x) \in V^{(p,q)}$, $w'(x) \in V^{(p',q')}$, we have $w(x) \otimes w'(x) \in V^{(p+p',q+q')}$ and thus $R^\rho{}_{\alpha\mu\nu} X^\alpha \in V^{(a_R,\, b_R+1)}$

If other way, symm. prop. would be different and $R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}$ defined differently etc.

(this is a contraction, not a tensor product)

where one index appears twice and can be contracted. The contraction then yields a tensorfield $C(R^{\rho}_{\alpha\mu\nu} X^{\alpha}) \in V^{(a_R-1, b_R)}$.
which has to be equivalent to the $X^{\rho}_{;\nu;\mu} - X^{\rho}_{;\mu;\nu} \in V^{(3,1)}$
tensorfield (the summation is, of course for $w(x) \in V^{(p,q)}$, $w(x) \in V^{(p,q)}$ given by a tensorfield $w(x+x) \in V^{(p,q)}$) and thus $a_R = 3, b_R = 1$
$\Rightarrow$ 1-times contravariant and 3-times covariant.

How can we sum tensors only for same rank tensors?
we can only sum same rank tensors defined for each space-time point. Componentwise

Alternatively: look at definitions of

$$R^{\rho}_{\alpha\mu\nu} = \partial_\mu \Gamma^{\rho}_{\alpha\nu} - \partial_\nu \Gamma^{\rho}_{\alpha\mu} + \Gamma^{\delta}_{\alpha\nu}\Gamma^{\rho}_{\delta\mu} - \Gamma^{\delta}_{\alpha\mu}\Gamma^{\rho}_{\delta\nu}$$

and argue in the same way: tensorproduct and contract, additionally, the derivative yields a $(p+1,q)$-tensorfield.

Which way better?
Second way not okay as $\nabla_\mu \neq \partial_\mu$ and thus we don't know that that is a $(p+1,q)$ tensorfield? Also we don't know that $\Gamma^{\rho}_{\alpha\nu}$ is a $(2,1)$-tensorfield?

True.

c) To see that $R(\varphi^* g) = \varphi^* R(g)$, we look at

$$R^{\rho}_{\alpha\mu\nu}(\varphi^* g) X^{\alpha} = X^{\rho}_{;\nu;\mu}(\varphi^* g) - X^{\rho}_{;\mu;\nu}(\varphi^* g)$$

?

Here you would have to use the transformat's and show explicitly

Apply φ^* to Γ or g which are in ∇X by definition?

d) $R_{\alpha\beta\mu\nu} := g_{\alpha\rho} R^{\rho}{}_{\beta\mu\nu}$

$$R_{\alpha\beta\mu\nu} = g_{\alpha\rho}\left\{ \frac{\partial}{\partial x^\mu}\Gamma^\rho_{\beta\nu} - \frac{\partial}{\partial x^\nu}\Gamma^\rho_{\beta\mu} + \Gamma^\delta_{\beta\nu}\Gamma^\rho_{\delta\mu} - \Gamma^\delta_{\beta\mu}\Gamma^\rho_{\delta\nu}\right\}$$

$$\frac{\partial}{\partial x^\mu}\left\{\frac{1}{2} g^{\rho\kappa}\left(\frac{\partial}{\partial x^\beta}g_{\kappa\nu} + \frac{\partial}{\partial x^\nu}g_{\kappa\beta} - \frac{\partial}{\partial x^\kappa}g_{\beta\nu}\right)\right\}$$

$$= \frac{1}{2}\left(\frac{\partial}{\partial x^\mu}g^{\rho\kappa}\right)\left(\frac{\partial}{\partial x^\beta}g_{\kappa\nu} + \frac{\partial}{\partial x^\nu}g_{\kappa\beta} - \frac{\partial}{\partial x^\kappa}g_{\beta\nu}\right)$$
$$+ \frac{1}{2} g^{\rho\kappa}\left(\frac{\partial^2}{\partial x^\mu \partial x^\beta}g_{\kappa\nu} + \frac{\partial^2}{\partial x^\mu \partial x^\nu}g_{\kappa\beta} - \frac{\partial^2}{\partial x^\mu \partial x^\kappa}g_{\beta\nu}\right)$$

$$\overset{g^{\rho\kappa}{}_{;\mu}=0}{=} \frac{1}{2}\left(-g^{\lambda\kappa}\Gamma^\rho_{\lambda\mu} - g^{\rho\lambda}\Gamma^\kappa_{\lambda\mu}\right)\left(\frac{\partial}{\partial x^\beta}g_{\kappa\nu} + \frac{\partial}{\partial x^\nu}g_{\kappa\beta} - \frac{\partial}{\partial x^\kappa}g_{\beta\nu}\right)$$
$$+ \frac{1}{2} g^{\rho\kappa}\left(\frac{\partial^2}{\partial x^\mu \partial x^\beta}g_{\kappa\nu} + \frac{\partial^2}{\partial x^\mu \partial x^\nu}g_{\kappa\beta} - \frac{\partial^2}{\partial x^\mu \partial x^\kappa}g_{\beta\nu}\right)$$

$$\overset{g_{\kappa\delta}\Gamma^\delta_{\beta\nu} = \Gamma^\rho_{\beta\nu} \text{ w/o } g^{\rho\kappa} \text{ in front}}{=} -\left(g^{\lambda\kappa}\Gamma^\rho_{\lambda\mu} + g^{\rho\lambda}\Gamma^\kappa_{\lambda\mu}\right) g_{\kappa\delta}\Gamma^\delta_{\beta\nu}$$
$$+ \frac{1}{2} g^{\rho\kappa}\left(\frac{\partial^2}{\partial x^\mu \partial x^\beta}g_{\kappa\nu} + \frac{\partial^2}{\partial x^\mu \partial x^\nu}g_{\kappa\beta} - \frac{\partial^2}{\partial x^\mu \partial x^\kappa}g_{\beta\nu}\right)$$

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Analogously:

$$\frac{\partial}{\partial x^\nu}\Gamma^\rho_{\beta\mu} = -\left(g^{\lambda\kappa}\Gamma^\rho_{\lambda\nu} + g^{\rho\lambda}\Gamma^\kappa_{\lambda\nu}\right) g_{\kappa\delta}\Gamma^\delta_{\beta\mu}$$
$$+ \frac{1}{2} g^{\rho\kappa}\left(\frac{\partial^2}{\partial x^\nu \partial x^\beta}g_{\kappa\mu} + \frac{\partial^2}{\partial x^\nu \partial x^\mu}g_{\kappa\beta} - \frac{\partial^2}{\partial x^\nu \partial x^\kappa}g_{\beta\mu}\right)$$

$$= g_{\alpha\rho}\left\{\frac{1}{2} g^{\rho\kappa}\left(\frac{\partial^2}{\partial x^\mu \partial x^\beta}g_{\kappa\nu} - \frac{\partial^2}{\partial x^\mu \partial x^\kappa}g_{\beta\nu} - \frac{\partial^2}{\partial x^\nu \partial x^\beta}g_{\kappa\mu} + \frac{\partial^2}{\partial x^\nu \partial x^\kappa}g_{\beta\mu}\right)\right.$$
$$-\left(g^{\lambda\kappa}\Gamma^\rho_{\lambda\mu} + g^{\rho\lambda}\Gamma^\kappa_{\lambda\mu}\right) g_{\kappa\delta}\Gamma^\delta_{\beta\nu}$$
$$\left.+\left(g^{\lambda\kappa}\Gamma^\rho_{\lambda\nu} + g^{\rho\lambda}\Gamma^\kappa_{\lambda\nu}\right) g_{\kappa\delta}\Gamma^\delta_{\beta\mu} + \Gamma^\delta_{\beta\nu}\Gamma^\rho_{\delta\mu} - \Gamma^\delta_{\beta\mu}\Gamma^\rho_{\delta\nu}\right\}$$

$$= \frac{1}{2}\left\{\frac{\partial^2}{\partial x^\mu \partial x^\beta}g_{\alpha\nu} - \frac{\partial^2}{\partial x^\mu \partial x^\alpha}g_{\beta\nu} - \frac{\partial^2}{\partial x^\nu \partial x^\beta}g_{\alpha\mu} + \frac{\partial^2}{\partial x^\nu \partial x^\alpha}g_{\beta\mu}\right\}$$
$$-\left(\delta^\lambda_\delta g_{\alpha\rho}\Gamma^\rho_{\lambda\mu} + \delta^\lambda_\alpha g_{\kappa\delta}\Gamma^\kappa_{\lambda\mu}\right)\Gamma^\delta_{\beta\nu}$$
$$+\left(\delta^\lambda_\delta g_{\alpha\rho}\Gamma^\rho_{\lambda\nu} + \delta^\lambda_\alpha g_{\kappa\delta}\Gamma^\kappa_{\lambda\nu}\right)\Gamma^\delta_{\beta\mu} + g_{\alpha\rho}\Gamma^\delta_{\beta\nu}\Gamma^\rho_{\delta\mu} - g_{\alpha\rho}\Gamma^\delta_{\beta\mu}\Gamma^\rho_{\delta\nu}\}$$

part. derivatives commute

$$= \frac{1}{2}\left\{\frac{\partial^2}{\partial x^\beta \partial x^\mu}g_{\alpha\nu} + \frac{\partial^2}{\partial x^\alpha \partial x^\nu}g_{\beta\mu} - \frac{\partial^2}{\partial x^\beta \partial x^\nu}g_{\alpha\mu} - \frac{\partial^2}{\partial x^\alpha \partial x^\mu}g_{\beta\nu}\right\}$$
$$- g_{\alpha\rho}\Gamma^\rho_{\delta\mu}\Gamma^\delta_{\beta\nu} - g_{\kappa\delta}\Gamma^\kappa_{\alpha\mu}\Gamma^\delta_{\beta\nu} + g_{\alpha\rho}\Gamma^\rho_{\delta\nu}\Gamma^\delta_{\beta\mu} + g_{\kappa\delta}\Gamma^\kappa_{\alpha\nu}\Gamma^\delta_{\beta\mu}$$
$$+ g_{\alpha\rho}\Gamma^\delta_{\beta\nu}\Gamma^\rho_{\delta\mu} - g_{\alpha\rho}\Gamma^\delta_{\beta\mu}\Gamma^\rho_{\delta\nu}$$

$$= \frac{1}{2}\left\{\frac{\partial^2}{\partial x^\beta \partial x^\mu}g_{\alpha\nu} + \frac{\partial^2}{\partial x^\alpha \partial x^\nu}g_{\beta\mu} - \frac{\partial^2}{\partial x^\beta \partial x^\nu}g_{\alpha\mu} - \frac{\partial^2}{\partial x^\alpha \partial x^\mu}g_{\beta\nu}\right\}$$
$$+ g_{\lambda\delta}\left\{\Gamma^\lambda_{\alpha\nu}\Gamma^\delta_{\beta\mu} - \Gamma^\lambda_{\alpha\mu}\Gamma^\delta_{\beta\nu}\right\}$$

$$\overset{\delta \leftrightarrow \lambda}{\underset{g\,\text{sym}}{=}} \frac{1}{2} \left\{ \frac{\partial^2}{\partial x^\beta \partial x^\mu} g_{\alpha\nu} + \frac{\partial^2}{\partial x^\alpha \partial x^\nu} g_{\beta\mu} - \frac{\partial^2}{\partial x^\beta \partial x^\nu} g_{\alpha\mu} - \frac{\partial^2}{\partial x^\alpha \partial x^\mu} g_{\beta\nu} \right\}$$

$$+ g_{\lambda\sigma} \left(\Gamma^\lambda_{\beta\mu} \Gamma^\sigma_{\alpha\nu} - \Gamma^\lambda_{\beta\nu} \Gamma^\sigma_{\alpha\mu} \right) \quad \square$$

where pretty much in the beginning we used

$$g^{\rho\kappa}{}_{;\mu} \overset{\#2a)}{=} 0 = \frac{\partial}{\partial x^\mu} g^{\rho\kappa} + g^{\lambda\kappa} \Gamma^\rho_{\lambda\mu} + g^{\rho\lambda} \Gamma^\kappa_{\lambda\mu}$$

and

$$g_{\kappa\sigma} \Gamma^\sigma_{\beta\nu} = g_{\kappa\sigma} \frac{1}{2} g^{\sigma\epsilon} \left(\frac{\partial}{\partial x^\beta} g_{\epsilon\nu} + \frac{\partial}{\partial x^\nu} g_{\epsilon\beta} - \frac{\partial}{\partial x^\epsilon} g_{\beta\nu} \right)$$

$$= \frac{1}{2} \left(\frac{\partial}{\partial x^\beta} g_{\kappa\nu} + \frac{\partial}{\partial x^\nu} g_{\kappa\beta} - \frac{\partial}{\partial x^\kappa} g_{\beta\nu} \right)$$

Better way to see this?

not really. :(

d) ALTERNATIVE **WRONG?** (Would this also be successfull at some point?!)

$$R_{\alpha\beta\mu\nu} := g_{\alpha\rho} R^{\rho}{}_{\beta\mu\nu}$$

$$R_{\alpha\beta\mu\nu} = g_{\alpha\rho} \left\{ \frac{\partial}{\partial x^\mu} \Gamma^\rho_{\beta\nu} - \frac{\partial}{\partial x^\nu} \Gamma^\rho_{\beta\mu} + \Gamma^\delta_{\beta\nu} \Gamma^\rho_{\delta\mu} - \Gamma^\delta_{\beta\mu} \Gamma^\rho_{\delta\nu} \right\}$$

$$\frac{\partial}{\partial x^\mu} \Gamma^\rho_{\beta\nu} = \frac{\partial}{\partial x^\mu} \left\{ \frac{1}{2} g^{\rho\kappa} \left(\frac{\partial}{\partial x^\beta} g_{\kappa\nu} + \frac{\partial}{\partial x^\nu} g_{\kappa\beta} - \frac{\partial}{\partial x^\kappa} g_{\beta\nu} \right) \right\}$$

$$= \frac{1}{2} \left(\frac{\partial}{\partial x^\mu} g^{\rho\kappa} \right) \left(\frac{\partial}{\partial x^\beta} g_{\kappa\nu} + \frac{\partial}{\partial x^\nu} g_{\kappa\beta} - \frac{\partial}{\partial x^\kappa} g_{\beta\nu} \right)$$
$$+ \frac{1}{2} g^{\rho\kappa} \left(\frac{\partial^2}{\partial x^\mu \partial x^\beta} g_{\kappa\nu} + \underbrace{\frac{\partial^2}{\partial x^\mu \partial x^\nu} g_{\kappa\beta}} - \frac{\partial^2}{\partial x^\mu \partial x^\kappa} g_{\beta\nu} \right)$$

$$\frac{\partial}{\partial x^\nu} \Gamma^\rho_{\beta\mu} = \frac{\partial}{\partial x^\nu} \left\{ \frac{1}{2} g^{\rho\kappa} \left(\frac{\partial}{\partial x^\beta} g_{\kappa\mu} + \frac{\partial}{\partial x^\mu} g_{\kappa\beta} - \frac{\partial}{\partial x^\kappa} g_{\beta\mu} \right) \right\}$$

$$= \frac{1}{2} \left(\frac{\partial}{\partial x^\nu} g^{\rho\kappa} \right) \left(\frac{\partial}{\partial x^\beta} g_{\kappa\mu} + \frac{\partial}{\partial x^\mu} g_{\kappa\beta} - \frac{\partial}{\partial x^\kappa} g_{\beta\mu} \right)$$
$$+ \frac{1}{2} g^{\rho\kappa} \left(\frac{\partial^2}{\partial x^\nu \partial x^\beta} g_{\kappa\mu} + \underbrace{\frac{\partial^2}{\partial x^\nu \partial x^\mu} g_{\kappa\beta}} - \frac{\partial^2}{\partial x^\nu \partial x^\kappa} g_{\beta\mu} \right)$$

$$= g_{\alpha\rho} \Big\{ \frac{1}{2} (\partial_\mu g^{\rho\kappa}) (\partial_\beta g_{\kappa\nu} + \partial_\nu g_{\kappa\beta} - \partial_\kappa g_{\beta\nu})$$
$$+ \frac{1}{2} g^{\rho\kappa} (\partial^2_{\mu\beta} g_{\kappa\nu} - \partial^2_{\mu\kappa} g_{\beta\nu}) - \frac{1}{2} g^{\rho\kappa} (\partial^2_{\nu\beta} g_{\kappa\mu} - \partial^2_{\nu\kappa} g_{\beta\mu})$$
$$- \frac{1}{2} (\partial_\nu g^{\rho\kappa}) (\partial_\beta g_{\kappa\mu} + \partial_\mu g_{\kappa\beta} - \partial_\kappa g_{\beta\mu})$$
$$+ \frac{1}{4} g^{\delta\kappa} g^{\rho\epsilon} (\partial_\beta g_{\kappa\nu} + \partial_\nu g_{\kappa\beta} - \partial_\kappa g_{\beta\nu}) (\partial_\delta g_{\epsilon\mu} + \partial_\mu g_{\epsilon\delta} - \partial_\epsilon g_{\delta\mu})$$
$$- \frac{1}{4} g^{\delta\kappa} g^{\rho\epsilon} (\partial_\beta g_{\kappa\mu} + \partial_\mu g_{\kappa\beta} - \partial_\kappa g_{\beta\mu}) (\partial_\delta g_{\epsilon\nu} + \partial_\nu g_{\epsilon\delta} - \partial_\epsilon g_{\delta\nu})$$

$$= \frac{1}{2} g_{\alpha\rho} (\partial_\mu g^{\rho\kappa}) (\partial_\beta g_{\kappa\nu} + \partial_\nu g_{\kappa\beta} - \partial_\kappa g_{\beta\nu})$$
$$+ \frac{1}{2} (\partial^2_{\mu\beta} g_{\alpha\nu} - \partial^2_{\mu\alpha} g_{\beta\nu}) - \frac{1}{2} (\partial^2_{\nu\beta} g_{\alpha\mu} - \partial^2_{\nu\alpha} g_{\beta\mu})$$
$$- \frac{1}{2} g_{\alpha\rho} (\partial_\nu g^{\rho\kappa}) (\partial_\beta g_{\kappa\mu} + \partial_\mu g_{\kappa\beta} - \partial_\kappa g_{\beta\mu})$$
$$+ \frac{1}{4} g^{\delta\kappa} (\partial_\beta g_{\kappa\nu} + \partial_\nu g_{\kappa\beta} - \partial_\kappa g_{\beta\nu}) (\partial_\delta g_{\alpha\mu} + \partial_\mu g_{\alpha\delta} - \partial_\alpha g_{\delta\mu})$$
$$- \frac{1}{4} g^{\delta\kappa} (\partial_\beta g_{\kappa\mu} + \partial_\mu g_{\kappa\beta} - \partial_\kappa g_{\beta\mu}) (\partial_\delta g_{\alpha\nu} + \partial_\nu g_{\alpha\delta} - \partial_\alpha g_{\delta\nu})$$

I can't follow. But it should work if you consider all terms.

$$= \frac{1}{2} \left\{ \partial^2_{\mu\beta} g_{\alpha\nu} + \partial^2_{\nu\alpha} g_{\beta\mu} - \partial^2_{\mu\alpha} g_{\beta\nu} - \partial^2_{\nu\beta} g_{\alpha\mu} \right\}$$
$$+ \frac{1}{2} g_{\alpha\rho} \Big\{ \partial_\mu g^{\rho\kappa} \partial_\beta g_{\kappa\nu} + \partial_\mu g^{\rho\kappa} \partial_\nu g_{\kappa\beta} - \partial_\mu g^{\rho\kappa} \partial_\kappa g_{\beta\nu}$$
$$- \partial_\nu g^{\rho\kappa} \partial_\beta g_{\kappa\mu} - \partial_\nu g^{\rho\kappa} \partial_\mu g_{\kappa\beta} + \partial_\nu g^{\rho\kappa} \partial_\kappa g_{\beta\mu} \Big\}$$
$$+ \frac{1}{4} g^{\delta\kappa} \Big\{ \partial_\beta g_{\kappa\nu} \partial_\delta g_{\alpha\mu} + \partial_\beta g_{\kappa\nu} \partial_\mu g_{\alpha\delta} - \partial_\beta g_{\kappa\nu} \partial_\alpha g_{\delta\mu} + \partial_\nu g_{\kappa\beta} \partial_\delta g_{\alpha\mu}$$
$$+ \partial_\nu g_{\kappa\beta} \partial_\mu g_{\alpha\delta} - \partial_\nu g_{\kappa\beta} \partial_\alpha g_{\delta\mu} - \partial_\kappa g_{\beta\nu} \partial_\delta g_{\alpha\mu} - \partial_\kappa g_{\beta\nu} \partial_\mu g_{\alpha\delta}$$
$$+ \partial_\kappa g \ldots \Big\}$$

$$-\frac{1}{4} g^{\delta\kappa}\{\partial_\beta g_{\kappa\mu}\partial_\delta g_{\alpha\nu} + \partial_\beta g_{\kappa\mu}\partial_\nu g_{\alpha\delta} - \partial_\beta g_{\kappa\mu}\partial_\alpha g_{\delta\nu} + \partial_\mu g_{\kappa\beta}\partial_\delta g_{\alpha\nu}$$
$$+ \partial_\mu g_{\kappa\beta}\partial_\nu g_{\alpha\delta} - \partial_\mu g_{\kappa\beta}\partial_\alpha g_{\delta\nu} - \partial_\kappa g_{\beta\mu}\partial_\delta g_{\alpha\nu} - \partial_\kappa g_{\beta\mu}\partial_\nu g_{\alpha\delta}$$
$$+ \partial_\kappa g_{\beta\mu}\partial_\alpha g_{\delta\nu}\}$$

$$g_{\alpha\rho}\partial_\mu g^{\rho\kappa} = \partial_\mu(g_{\alpha\rho}g^{\rho\kappa}) - g^{\rho\kappa}\partial_\mu g_{\alpha\rho} = -g^{\rho\kappa}\partial_\mu g_{\alpha\rho} \text{ etc.}$$

$$= \frac{1}{2}\{\partial^2_{\mu\beta} g_{\alpha\nu} + \partial^2_{\nu\alpha} g_{\beta\mu} - \partial^2_{\mu\alpha} g_{\beta\nu} - \partial^2_{\nu\beta} g_{\alpha\mu}\}$$
$$-\frac{1}{2} g^{\rho\kappa}\{\partial_\mu g_{\alpha\rho}\partial_\beta g_{\kappa\nu} + \partial_\mu g_{\alpha\rho}\partial_\nu g_{\kappa\beta} - \partial_\mu g_{\alpha\rho}\partial_\kappa g_{\beta\nu}$$
$$-\partial_\nu g_{\alpha\rho}\partial_\beta g_{\kappa\mu} - \partial_\nu g_{\alpha\rho}\partial_\mu g_{\kappa\beta} + \partial_\nu g_{\alpha\rho}\partial_\kappa g_{\beta\mu}\}$$
$$+\frac{1}{4} g^{\delta\kappa}\{\partial_\beta g_{\kappa\nu}\partial_\delta g_{\alpha\mu} + \partial_\beta g_{\kappa\nu}\partial_\mu g_{\alpha\delta} - \partial_\beta g_{\kappa\nu}\partial_\alpha g_{\delta\mu} + \partial_\nu g_{\kappa\beta}\partial_\delta g_{\alpha\mu}$$
$$+\partial_\nu g_{\kappa\beta}\partial_\mu g_{\alpha\delta} - \partial_\nu g_{\kappa\beta}\partial_\alpha g_{\delta\mu} - \partial_\kappa g_{\beta\nu}\partial_\delta g_{\alpha\mu} - \partial_\kappa g_{\beta\nu}\partial_\mu g_{\alpha\delta}$$
$$+\partial_\kappa g_{\beta\nu}\partial_\alpha g_{\delta\mu}\}$$
$$-\frac{1}{4} g^{\delta\kappa}\{\partial_\beta g_{\kappa\mu}\partial_\delta g_{\alpha\nu} + \partial_\beta g_{\kappa\mu}\partial_\nu g_{\alpha\delta} - \partial_\beta g_{\kappa\mu}\partial_\alpha g_{\delta\nu} + \partial_\mu g_{\kappa\beta}\partial_\delta g_{\alpha\nu}$$
$$+\partial_\mu g_{\kappa\beta}\partial_\nu g_{\alpha\delta} - \partial_\mu g_{\kappa\beta}\partial_\alpha g_{\delta\nu} - \partial_\kappa g_{\beta\mu}\partial_\delta g_{\alpha\nu} - \partial_\kappa g_{\beta\mu}\partial_\nu g_{\alpha\delta}$$
$$+\partial_\kappa g_{\beta\mu}\partial_\alpha g_{\delta\nu}\}$$

$$= \frac{1}{2}\{\partial^2_{\mu\beta} g_{\alpha\nu} + \partial^2_{\nu\alpha} g_{\beta\mu} - \partial^2_{\mu\alpha} g_{\beta\nu} - \partial^2_{\nu\beta} g_{\alpha\mu}\}$$
$$-\frac{1}{4} g^{\rho\kappa}\{2\partial_\mu g_{\alpha\rho}\partial_\beta g_{\kappa\nu} + 2\partial_\mu g_{\alpha\rho}\partial_\nu g_{\kappa\beta} - 2\partial_\mu g_{\alpha\rho}\partial_\kappa g_{\beta\nu}$$
$$-2\partial_\nu g_{\alpha\rho}\partial_\beta g_{\kappa\mu} - 2\partial_\nu g_{\alpha\rho}\partial_\mu g_{\kappa\beta} + 2\partial_\nu g_{\alpha\rho}\partial_\kappa g_{\beta\mu}$$
$$-\partial_\beta g_{\kappa\nu}\partial_\rho g_{\alpha\mu} - \partial_\beta g_{\kappa\nu}\partial_\mu g_{\alpha\rho} + \partial_\beta g_{\kappa\nu}\partial_\alpha g_{\rho\mu} - \partial_\nu g_{\kappa\beta}\partial_\rho g_{\alpha\mu}$$
$$-\partial_\nu g_{\kappa\beta}\partial_\mu g_{\alpha\rho} + \partial_\nu g_{\kappa\beta}\partial_\alpha g_{\rho\mu} + \partial_\kappa g_{\beta\nu}\partial_\rho g_{\alpha\mu} + \partial_\kappa g_{\beta\nu}\partial_\mu g_{\alpha\rho}$$
$$-\partial_\kappa g_{\beta\nu}\partial_\alpha g_{\rho\mu}$$
$$+\partial_\beta g_{\kappa\mu}\partial_\rho g_{\alpha\nu} + \partial_\beta g_{\kappa\mu}\partial_\nu g_{\alpha\rho} - \partial_\beta g_{\kappa\mu}\partial_\alpha g_{\rho\nu} + \partial_\mu g_{\kappa\beta}\partial_\rho g_{\alpha\nu}$$
$$+\partial_\mu g_{\kappa\beta}\partial_\nu g_{\alpha\rho} - \partial_\mu g_{\kappa\beta}\partial_\alpha g_{\rho\nu} - \partial_\kappa g_{\beta\mu}\partial_\rho g_{\alpha\nu} - \partial_\kappa g_{\beta\mu}\partial_\nu g_{\alpha\rho}$$
$$+\partial_\kappa g_{\beta\mu}\partial_\alpha g_{\rho\nu}\}$$

$$= \frac{1}{2}\{\partial^2_{\mu\beta} g_{\alpha\nu} + \partial^2_{\nu\alpha} g_{\beta\mu} - \partial^2_{\mu\alpha} g_{\beta\nu} - \partial^2_{\nu\beta} g_{\alpha\mu}\}$$
$$-\frac{1}{4} g^{\rho\kappa}\{\partial_\mu g_{\alpha\rho}\partial_\beta g_{\kappa\nu} + \partial_\mu g_{\alpha\rho}\partial_\nu g_{\kappa\beta} - \partial_\mu g_{\alpha\rho}\partial_\kappa g_{\beta\nu} - \partial_\nu g_{\alpha\rho}\partial_\beta g_{\kappa\mu}$$
$$-\partial_\nu g_{\alpha\rho}\partial_\mu g_{\kappa\beta} + \partial_\nu g_{\alpha\rho}\partial_\kappa g_{\beta\mu}\}$$
$$+\frac{1}{4} g^{\rho\kappa}(\partial_\rho g_{\alpha\mu})(\partial_\beta g_{\kappa\nu} + \partial_\nu g_{\kappa\beta} - \partial_\kappa g_{\beta\nu}) - \frac{1}{4} g^{\rho\kappa}(\partial_\alpha g_{\rho\mu})(\partial_\beta g_{\kappa\nu} + \partial_\nu g_{\kappa\beta} - \partial_\kappa g_{\beta\nu})$$
$$-\frac{1}{4} g^{\rho\kappa}(\partial_\rho g_{\alpha\nu})(\partial_\beta g_{\kappa\mu} + \partial_\mu g_{\kappa\beta} - \partial_\kappa g_{\beta\mu}) + \frac{1}{4} g^{\rho\kappa}(\partial_\alpha g_{\rho\nu})(\partial_\beta g_{\kappa\mu} + \partial_\mu g_{\kappa\beta} - \partial_\kappa g_{\beta\mu})$$

e) $R_{\alpha\beta\mu\nu} = \frac{1}{2}\left(\partial_\beta\partial_\mu g_{\alpha\nu} + \partial_\alpha\partial_\nu g_{\beta\mu} - \partial_\beta\partial_\nu g_{\alpha\mu} - \partial_\alpha\partial_\mu g_{\beta\nu}\right)$
$+ g_{\lambda\delta}\left(\Gamma^\lambda_{\beta\mu}\Gamma^\delta_{\alpha\nu} - \Gamma^\lambda_{\beta\nu}\Gamma^\delta_{\alpha\mu}\right)$

$\overset{\text{reorder}}{=} \frac{1}{2}\left(-\partial_\beta\partial_\nu g_{\alpha\mu} - \partial_\alpha\partial_\mu g_{\beta\nu} + \partial_\beta\partial_\mu g_{\alpha\nu} + \partial_\alpha\partial_\nu g_{\beta\mu}\right)$
$+ g_{\lambda\delta}\left(-\Gamma^\lambda_{\beta\nu}\Gamma^\delta_{\alpha\mu} + \Gamma^\lambda_{\beta\mu}\Gamma^\delta_{\alpha\nu}\right)$

2

$= -\Big\{\frac{1}{2}\left(\partial_\beta\partial_\nu g_{\alpha\mu} + \partial_\alpha\partial_\mu g_{\beta\nu} - \partial_\beta\partial_\mu g_{\alpha\nu} - \partial_\alpha\partial_\nu g_{\beta\mu}\right)$
$+ g_{\lambda\delta}\left(\Gamma^\lambda_{\beta\nu}\Gamma^\delta_{\alpha\mu} - \Gamma^\lambda_{\beta\mu}\Gamma^\delta_{\alpha\nu}\right)\Big\}$

$= -R_{\alpha\beta\nu\mu}$

$R_{\alpha\beta\mu\nu} \overset{\text{reorder}}{=} \frac{1}{2}\left(-\partial_\alpha\partial_\mu g_{\beta\nu} - \partial_\beta\partial_\nu g_{\alpha\mu} + \partial_\alpha\partial_\nu g_{\beta\mu} + \partial_\beta\partial_\mu g_{\alpha\nu}\right)$
$+ g_{\lambda\delta}\left(-\Gamma^\delta_{\alpha\mu}\Gamma^\lambda_{\beta\nu} + \Gamma^\delta_{\alpha\nu}\Gamma^\lambda_{\beta\mu}\right)$

$= -\Big\{\frac{1}{2}\left(\partial_\alpha\partial_\mu g_{\beta\nu} + \partial_\beta\partial_\nu g_{\alpha\mu} - \partial_\alpha\partial_\nu g_{\beta\mu} - \partial_\beta\partial_\mu g_{\alpha\nu}\right)$
$+ g_{\lambda\delta}\left(\Gamma^\delta_{\alpha\mu}\Gamma^\lambda_{\beta\nu} - \Gamma^\delta_{\alpha\nu}\Gamma^\lambda_{\beta\mu}\right)\Big\}$

$\overset{\delta\leftrightarrow\lambda \text{ and } g \text{ sym.}}{=} -R_{\beta\alpha\mu\nu}$

$R_{\alpha\beta\mu\nu} \overset{\Gamma\text{ sym.}}{=} \frac{1}{2}\left(\partial_\alpha\partial_\nu g_{\beta\mu} + \partial_\beta\partial_\mu g_{\alpha\nu} - \partial_\beta\partial_\nu g_{\alpha\mu} - \partial_\alpha\partial_\mu g_{\beta\nu}\right)$
$+ g_{\lambda\delta}\left(\Gamma^\delta_{\nu\alpha}\Gamma^\lambda_{\mu\beta} - \Gamma^\lambda_{\nu\beta}\Gamma^\delta_{\mu\alpha}\right)$

$\overset{\partial\text{'s commute, } g\text{ sym., } \delta\leftrightarrow\lambda \text{ for first part}}{=} \frac{1}{2}\left(\partial_\nu\partial_\alpha g_{\mu\beta} + \partial_\mu\partial_\beta g_{\nu\alpha} - \partial_\nu\partial_\beta g_{\mu\alpha} - \partial_\mu\partial_\alpha g_{\nu\beta}\right)$
$+ g_{\lambda\delta}\left(\Gamma^\lambda_{\nu\alpha}\Gamma^\delta_{\mu\beta} - \Gamma^\lambda_{\nu\beta}\Gamma^\delta_{\mu\alpha}\right)$

$= R_{\mu\nu\alpha\beta}$

I don't know :(
↓
Faster way to see this?

f) Claim: $R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} = 0$

$R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta}$

1

$= \frac{1}{2}\left\{\partial_\beta\partial_\mu g_{\alpha\nu} + \partial_\alpha\partial_\nu g_{\beta\mu} - \partial_\beta\partial_\nu g_{\alpha\mu} - \partial_\alpha\partial_\mu g_{\beta\nu}\right\} + g_{\lambda\delta}\left(\Gamma^\lambda_{\beta\mu}\Gamma^\delta_{\alpha\nu} - \Gamma^\lambda_{\beta\nu}\Gamma^\delta_{\alpha\mu}\right)$
$+ \frac{1}{2}\left\{\partial_\nu\partial_\beta g_{\alpha\mu} + \partial_\alpha\partial_\mu g_{\nu\beta} - \partial_\nu\partial_\mu g_{\alpha\beta} - \partial_\alpha\partial_\beta g_{\nu\mu}\right\} + g_{\lambda\delta}\left(\Gamma^\lambda_{\nu\beta}\Gamma^\delta_{\alpha\mu} - \Gamma^\lambda_{\nu\mu}\Gamma^\delta_{\alpha\beta}\right)$
$+ \frac{1}{2}\left\{\partial_\mu\partial_\nu g_{\alpha\beta} + \partial_\alpha\partial_\beta g_{\mu\nu} - \partial_\mu\partial_\beta g_{\alpha\nu} - \partial_\alpha\partial_\nu g_{\mu\beta}\right\} + g_{\lambda\delta}\left(\Gamma^\lambda_{\mu\nu}\Gamma^\delta_{\alpha\beta} - \Gamma^\lambda_{\mu\beta}\Gamma^\delta_{\alpha\nu}\right)$

$= 0$