Disclaimer

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https://www.physics-and-stuff.com/

I raise no claim to correctness and completeness of the given solutions! This equally applies to the corrections mentioned above.

30.04.2018 H6) a) want to prove : $R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\lambda\mu;\nu} + R_{\alpha\beta\nu\lambda;\mu} \stackrel{!}{=} 0$ 5/5

We recall that e.g.

$$R_{\alpha\beta\mu\nu;\lambda} = \frac{\partial}{\partial x^\lambda} R_{\alpha\beta\mu\nu} - R_{\epsilon\beta\mu\nu}\Gamma^\epsilon_{\alpha\lambda} - R_{\alpha\epsilon\mu\nu}\Gamma^\epsilon_{\beta\lambda} - R_{\alpha\beta\epsilon\nu}\Gamma^\epsilon_{\mu\lambda} - R_{\alpha\beta\mu\epsilon}\Gamma^\epsilon_{\nu\lambda}$$

where we left out the explicit x dependence of the Riemann-tensor $R_{\alpha\beta\mu\nu}(x)$.

Why is this locally inertial coord. system valid? Isn't it just in normal/Riemann Coord. like this? 2

We now transform to a locally inertial coordinate system for the fixed x, we want to prove the identity for. this will then be valid for all x, as x is arbitrary. In this frame, the Christoffel-symbols vanish at the origin, i.e. $\Gamma^\alpha_{\beta\delta}(0) = 0$, while the derivative does not necessarily have to vanish, $\Gamma^\alpha_{\beta\delta,\delta}(0) \in$

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What is it pictorially where $\Gamma(0)=0$? And why not possible to also have $\Gamma^\alpha_{\beta\delta,\sigma}=0$?

Because you can make a coordinate transform in which g and ∂g vanishes

But you cannot find one in which $g, \partial g, \partial\partial g$ all vanish, $\therefore \partial\Gamma \neq 0$.

It then suffices to prove $R_{\alpha\beta\mu\nu,\lambda} + R_{\alpha\beta\lambda\mu,\nu} + R_{\alpha\beta\nu\lambda,\mu} \stackrel{!}{=} 0$ as all other terms vanish. We will give two proofs for this:

First: $\frac{\partial}{\partial x^\lambda} R_{\alpha\beta\mu\nu} + \frac{\partial}{\partial x^\nu} R_{\alpha\beta\lambda\mu} + \frac{\partial}{\partial x^\mu} R_{\alpha\beta\nu\lambda}$

$$= \frac{\partial}{\partial x^\lambda}\left(g_{\alpha\epsilon} R^\epsilon_{\ \beta\mu\nu}\right) + \frac{\partial}{\partial x^\nu}\left(g_{\alpha\epsilon} R^\epsilon_{\ \beta\lambda\mu}\right) + \frac{\partial}{\partial x^\mu}\left(g_{\alpha\epsilon} R^\epsilon_{\ \beta\nu\lambda}\right)$$

$$\left| \; R^\alpha_{\ \beta\mu\nu} = \frac{\partial}{\partial x^\mu}\Gamma^\alpha_{\beta\nu} - \frac{\partial}{\partial x^\nu}\Gamma^\alpha_{\beta\mu} + \Gamma^\alpha_{\lambda\mu}\Gamma^\lambda_{\beta\nu} - \Gamma^\alpha_{\lambda\nu}\Gamma^\lambda_{\beta\mu} \right.$$

$$\left| \; \stackrel{\text{locally inertial coord.}}{=} \frac{\partial}{\partial x^\mu}\Gamma^\alpha_{\beta\nu} - \frac{\partial}{\partial x^\nu}\Gamma^\alpha_{\beta\mu} \right.$$

$$= g_{\alpha\epsilon,\lambda} R^\epsilon_{\ \beta\mu\nu} + g_{\alpha\epsilon,\nu} R^\epsilon_{\ \beta\lambda\mu} + g_{\alpha\epsilon,\mu} R^\epsilon_{\ \beta\nu\lambda}$$

$$+ g_{\alpha\epsilon}\left\{ \frac{\partial}{\partial x^\lambda}\left(\frac{\partial}{\partial x^\mu}\Gamma^\epsilon_{\beta\nu} - \frac{\partial}{\partial x^\nu}\Gamma^\epsilon_{\beta\mu}\right) + \frac{\partial}{\partial x^\nu}\left(\frac{\partial}{\partial x^\lambda}\Gamma^\epsilon_{\beta\mu} - \frac{\partial}{\partial x^\mu}\Gamma^\epsilon_{\beta\lambda}\right) + \frac{\partial}{\partial x^\mu}\left(\frac{\partial}{\partial x^\nu}\Gamma^\epsilon_{\beta\lambda} - \frac{\partial}{\partial x^\lambda}\Gamma^\epsilon_{\beta\nu}\right)\right\}$$

$$= g_{\alpha\epsilon,\lambda} R^\epsilon_{\ \beta\mu\nu} + g_{\alpha\epsilon,\nu} R^\epsilon_{\ \beta\lambda\mu} + g_{\alpha\epsilon,\mu} R^\epsilon_{\ \beta\nu\lambda}$$

$$\left| \; g_{\alpha\beta;\mu} = \frac{\partial}{\partial x^\mu} g_{\alpha\beta} - g_{\epsilon\beta}\Gamma^\epsilon_{\alpha\mu} - g_{\alpha\epsilon}\Gamma^\epsilon_{\beta\mu} \stackrel{\text{sheet 1}}{=} 0 \stackrel{\text{inertial coord.}}{=} \frac{\partial}{\partial x^\mu} g_{\alpha\beta} = g_{\alpha\beta,\mu} \right.$$

$$= 0$$

The second proof uses the result of (45d), namely:

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}\left\{\frac{\partial^2}{\partial x^\beta \partial x^\mu} g_{\alpha\nu} + \frac{\partial^2}{\partial x^\alpha \partial x^\nu} g_{\beta\mu} - \frac{\partial^2}{\partial x^\beta \partial x^\nu} g_{\alpha\mu} - \frac{\partial^2}{\partial x^\alpha \partial x^\mu} g_{\beta\nu}\right\}$$

$$+ g_{\lambda\sigma}\underbrace{\left\{\Gamma^\lambda_{\beta\mu}\Gamma^\sigma_{\alpha\nu} - \Gamma^\lambda_{\beta\nu}\Gamma^\sigma_{\alpha\mu}\right\}}_{=0 \text{ in locally inertial coord. system}}$$

$$\Rightarrow \frac{\partial}{\partial x^\lambda} R_{\alpha\beta\mu\nu} + \frac{\partial}{\partial x^\nu} R_{\alpha\beta\lambda\mu} + \frac{\partial}{\partial x^\mu} R_{\alpha\beta\nu\lambda}$$

$$= \frac{1}{2}\Big\{\frac{\partial}{\partial x^\lambda}\Big(\frac{\partial^2}{\partial x^\beta \partial x^\mu} g_{\alpha\nu} + \frac{\partial^2}{\partial x^\alpha \partial x^\nu} g_{\beta\mu} - \frac{\partial^2}{\partial x^\beta \partial x^\nu} g_{\alpha\mu} - \frac{\partial^2}{\partial x^\alpha \partial x^\mu} g_{\beta\nu}\Big)$$

$$+ \frac{\partial}{\partial x^\nu}\Big(\frac{\partial^2}{\partial x^\beta \partial x^\lambda} g_{\alpha\mu} + \frac{\partial^2}{\partial x^\alpha \partial x^\mu} g_{\beta\lambda} - \frac{\partial^2}{\partial x^\beta \partial x^\mu} g_{\alpha\lambda} - \frac{\partial^2}{\partial x^\alpha \partial x^\lambda} g_{\beta\mu}\Big)$$

$$+ \frac{\partial}{\partial x^\mu}\Big(\frac{\partial^2}{\partial x^\beta \partial x^\nu} g_{\alpha\lambda} + \frac{\partial^2}{\partial x^\alpha \partial x^\lambda} g_{\beta\nu} - \frac{\partial^2}{\partial x^\beta \partial x^\lambda} g_{\alpha\nu} - \frac{\partial^2}{\partial x^\alpha \partial x^\nu} g_{\beta\lambda}\Big)\Big\}$$

$$= \frac{1}{2}\Big\{\frac{\partial^3}{\partial x^\lambda \partial x^\beta \partial x^\mu} g_{\alpha\nu} + \frac{\partial^3}{\partial x^\lambda \partial x^\alpha \partial x^\nu} g_{\beta\mu} - \frac{\partial^3}{\partial x^\lambda \partial x^\beta \partial x^\nu} g_{\alpha\mu} - \frac{\partial^3}{\partial x^\lambda \partial x^\alpha \partial x^\mu} g_{\beta\nu}$$

$$+ \frac{\partial^3}{\partial x^\nu \partial x^\beta \partial x^\lambda} g_{\alpha\mu} + \frac{\partial^3}{\partial x^\nu \partial x^\alpha \partial x^\mu} g_{\beta\lambda} - \frac{\partial^3}{\partial x^\nu \partial x^\beta \partial x^\mu} g_{\alpha\lambda} - \frac{\partial^3}{\partial x^\nu \partial x^\alpha \partial x^\lambda} g_{\beta\mu}$$

$$+ \frac{\partial^3}{\partial x^\mu \partial x^\beta \partial x^\nu} g_{\alpha\lambda} + \frac{\partial^3}{\partial x^\mu \partial x^\alpha \partial x^\lambda} g_{\beta\nu} - \frac{\partial^3}{\partial x^\mu \partial x^\beta \partial x^\lambda} g_{\alpha\nu} - \frac{\partial^3}{\partial x^\mu \partial x^\alpha \partial x^\nu} g_{\beta\lambda}\Big\}$$

$$= 0$$

Both proofs correct?

Yes!

But when you take derivative, you will have $\partial\Gamma$ terms. But only with a Γ term. ie $\partial(\Gamma\Gamma) = (\partial\Gamma)\underset{=0}{\Gamma} + \underset{=0}{\Gamma}(\partial\Gamma)$.

? But already set to zero before? If I had $\partial\Gamma$, then $\partial\Gamma \neq 0$?

therefore this term will not appear in the derivative.

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b) Equation (7) states $G^{\mu\nu}{}_{;\mu} = 0$, where

$$G^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} R_{\alpha\beta} - \frac{1}{2} g^{\mu\nu} R$$

$$R_{\alpha\beta} = R^{\kappa}{}_{\alpha\kappa\beta} \quad \text{and} \quad R = g^{\beta\kappa} R_{\beta\kappa}$$

Sheet 2: "all the other elem. tensor op. commute w/ ∇" → But raising/lowering only iff $\nabla g = 0$ for g the metric?

Yes, ~~But~~ ~~commutes~~ ~~$\Rightarrow \nabla g = 0$~~

Using that for the covariant derivative $(\nabla \omega)^{\mu_1 \dots \mu_l}{}_{\nu_1 \dots \nu_k \mu} \equiv \omega^{\mu_1 \dots \mu_l}{}_{\nu_1 \dots \nu_k ; \mu}$, we have the product rule $\nabla(\omega \otimes \omega') = (\nabla \omega) \otimes \omega' + \omega \otimes (\nabla \omega')$ and thus will pull the metric inside covariant derivatives in this exercise, making use of $0 = g_{\alpha\beta;\mu} = (\nabla g)_{\alpha\beta\mu}$. We will give two proofs for this:

?

Change back to other coord. w/ covariant derivative now, i.e. $\Gamma(0) \neq 0$?

Yes. Just a coordinate transfo. $R'(y) = \frac{\partial y}{\partial x} \frac{\partial}{\partial} \frac{\partial}{\partial} \frac{\partial}{\partial} R(x)$, $R(x) = 0 \Rightarrow R'(y) = 0$.

$g_{\alpha\beta;\alpha} = 0$ as well or only for different indices?!

$g_{\alpha\beta;\alpha}$ doesn't exist. Never have same index at low.

But $g^{\alpha}{}_{\beta;\alpha} = 0$

Yes.

First: from a), we know the differential BIANCHI identity:

$$R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\lambda\mu;\nu} + R_{\alpha\beta\nu\lambda;\mu} = 0 \quad \Big| \times g^{\alpha\mu} \text{ (summation included)}$$

$$\Leftrightarrow R^{\mu}{}_{\beta\mu\nu;\lambda} - R^{\mu}{}_{\beta\mu\lambda;\nu} + g^{\alpha\mu} R_{\alpha\beta\nu\lambda;\mu} = 0$$

where we used $R^{\mu}{}_{\beta\mu\nu;\lambda} = g^{\alpha\mu} R_{\alpha\beta\mu\nu;\lambda}$ and the (anti-)sym. of $R_{\alpha\beta\mu\nu}$

$$\Leftrightarrow R_{\beta\nu;\lambda} - R_{\beta\lambda;\nu} + g^{\alpha\mu} R_{\alpha\beta\nu\lambda;\mu} = 0 \quad \Big| \times g^{\beta\lambda} \text{ (" — ")}$$

$$\Leftrightarrow g^{\beta\lambda} R_{\beta\nu;\lambda} - R_{;\nu} + g^{\alpha\mu} R^{\lambda}{}_{\alpha\lambda\nu;\mu} = 0$$

using $R = g^{\beta\lambda} R_{\beta\lambda}$

$$\Leftrightarrow g^{\beta\lambda} R_{\beta\nu;\lambda} - R_{;\nu} + g^{\beta\lambda} R_{\beta\nu;\lambda} = 0$$

by renaming $\alpha \to \beta$, $\mu \to \lambda$ in last term

$$\Leftrightarrow g^{\beta\lambda} R_{\beta\nu;\lambda} = \frac{1}{2} R_{;\nu} \quad \Big| \times g^{\nu\kappa}$$

$$\Leftrightarrow g^{\lambda\beta} g^{\kappa\nu} R_{\beta\nu;\lambda} = \frac{1}{2} g^{\nu\kappa} R_{;\nu} \quad \Big| \text{ rename } \nu \leftrightarrow \lambda \text{ on r.h.s.}$$

$$\Leftrightarrow g^{\lambda\beta} g^{\kappa\nu} R_{\beta\nu;\lambda} = \frac{1}{2} g^{\lambda\kappa} R_{;\lambda}$$

$$\Leftrightarrow \left(g^{\lambda\beta} g^{\kappa\nu} R_{\beta\nu} - \frac{1}{2} g^{\lambda\kappa} R\right)_{;\lambda} = 0$$

$$\Leftrightarrow G^{\lambda\kappa}{}_{;\lambda} = 0$$

Alternative (easier?!) proof:

$$G^{\mu\nu}{}_{;\mu} = \left(g^{\mu\alpha} g^{\nu\beta} R_{\alpha\beta} - \frac{1}{2} g^{\mu\nu} R\right)_{;\mu}$$

$$= g^{\mu\alpha} g^{\nu\beta} R_{\alpha\beta;\mu} - \frac{1}{2} g^{\mu\nu} R_{;\mu}$$

$$= g^{\mu\alpha} g^{\nu\beta} R_{\alpha\beta;\mu} - \frac{1}{2} g^{\mu\nu} g^{\beta\lambda} R_{\beta\lambda;\mu}$$

$$\left| -\frac{1}{2} g^{\mu\nu} g^{\beta\lambda} R_{\beta\lambda;\mu} = -\frac{1}{2} g^{\mu\nu} g^{\beta\lambda} R^{\epsilon}{}_{\beta\epsilon\lambda;\mu} = -\frac{1}{2} g^{\mu\nu} g^{\beta\lambda} g^{\epsilon\delta} R_{\delta\beta\epsilon\lambda;\mu}\right.$$

$$\left| = -\frac{1}{2} g^{\mu\nu} g^{\beta\lambda} g^{\epsilon\delta} \left(-R_{\delta\beta\mu\epsilon;\lambda} - R_{\delta\beta\lambda\mu;\epsilon}\right)\right.$$

(anti-)sym of Riemann $= -\frac{1}{2} g^{\mu\nu} g^{\beta\lambda} R^{\epsilon}{}_{\beta\epsilon\mu;\lambda} - \frac{1}{2} g^{\mu\nu} g^{\epsilon\delta} R^{\lambda}{}_{\delta\lambda\mu;\epsilon}$

$$\left| = -\frac{1}{2} g^{\mu\nu} \left\{ g^{\beta\lambda} R_{\beta\mu;\lambda} + g^{\epsilon\delta} R_{\delta\mu;\epsilon} \right\}\right.$$

rename $= - g^{\mu\nu} g^{\beta\lambda} R_{\beta\mu;\lambda}$

$$= g^{\mu\alpha} g^{\nu\beta} R_{\alpha\beta;\mu} - g^{\mu\nu} g^{\beta\lambda} R_{\beta\mu;\lambda}$$

| rename for r.h.s.: $\beta \to \alpha$, $\lambda \to \mu$, $\mu \to \beta$

$$= g^{\mu\alpha} g^{\nu\beta} R_{\alpha\beta;\mu} - g^{\beta\nu} g^{\alpha\mu} R_{\alpha\beta;\mu} = 0$$