Disclaimer

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General Relativity Home exercise 4 — Marvin Zanke

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H7 X^μ a vectorfield, $\dot{y}^\mu(\tau) = X^\mu(y(\tau))$, solution uniquely given $y^\mu(0) = x^\mu$ and denoted $y(x,\tau)$.

$$f_\tau(x) = y(x,\tau) \rightsquigarrow f_\tau^{-1}(y(x,\tau)) = x = y(0)$$

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Why $f_\tau(x)$ diffeomorphisms? Fixed τ or x? $f_\tau : x \to y(x,\tau)$

a)
$$\frac{d}{d\tau} f_\tau(z)\Big|_{z=f_\tau^{-1}(y)} = \frac{d}{d\tau} y(z,\tau)\Big|_{z=f_\tau^{-1}(y)}$$

$$= \dot{y}(\underbrace{f_\tau^{-1}(y)}_{=x},\tau) = \dot{y}(\underset{x}{\underset{\shortparallel}{y(0)}},\tau) = X(y(\tau))$$

$\dot{y}$ indep. of $y(0)$

x-dependance in $y(x,\tau)$ only through initial cond.? If we have $\dot{y}(y(0),\tau)$, can then drop $y(0)$ dependence as for any init cond. the derivative is the same?!

Additionally $f_0(x) = y(x,0) = y(y(0),0) = y(0) = x$

$$\rightsquigarrow f_0 = \mathbb{1}$$

b)
$$\frac{d}{d\tau} f_\tau^* g_{\mu\nu}\Big|_{\tau=0} = \frac{d}{d\tau}\left\{ g_{\alpha\beta}(f_\tau(x)) \frac{\partial f_\tau^\alpha(x)}{\partial x^\mu} \frac{\partial f_\tau^\beta(x)}{\partial x^\nu}\right\}\Big|_{\tau=0}$$

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$$X_{\mu;\nu} + X_{\nu;\mu} = \frac{\partial}{\partial x^\nu} X_\mu - X_\alpha \Gamma^\alpha_{\mu\nu} + \frac{\partial}{\partial x^\mu} X_\nu - X_\alpha \Gamma^\alpha_{\nu\mu}$$

$$\overset{\Gamma\,\mathrm{sym}}{=} \frac{\partial}{\partial x^\nu} X_\mu + \frac{\partial}{\partial x^\mu} X_\nu - X_\alpha g^{\alpha\delta}\left(\frac{\partial}{\partial x^\mu} g_{\delta\nu} + \frac{\partial}{\partial x^\nu} g_{\delta\mu} - \frac{\partial}{\partial x^\delta} g_{\mu\nu}\right)$$

$$= \frac{\partial}{\partial x^\nu} X_\mu + \frac{\partial}{\partial x^\mu} X_\nu - X^\delta \left\{\frac{\partial}{\partial x^\mu} g_{\delta\nu} + \frac{\partial}{\partial x^\nu} g_{\delta\mu} - \frac{\partial}{\partial x^\delta} g_{\mu\nu}\right\}$$

$$= \frac{\partial}{\partial x^\nu} X_\mu + \frac{\partial}{\partial x^\mu} X_\nu - \left\{\frac{\partial}{\partial x^\mu}(X^\delta g_{\delta\nu}) - g_{\delta\nu}\left(\frac{\partial}{\partial x^\mu} X^\delta\right) + \frac{\partial}{\partial x^\nu}(X^\delta g_{\delta\mu}) - g_{\delta\mu}\left(\frac{\partial}{\partial x^\nu} X^\delta\right)\right\} + X^\delta \frac{\partial}{\partial x^\delta} g_{\mu\nu}$$

$$= g_{\delta\nu} \frac{\partial}{\partial x^\mu} X^\delta + g_{\delta\mu} \frac{\partial}{\partial x^\nu} X^\delta + X^\delta \frac{\partial}{\partial x^\delta} g_{\mu\nu} \qquad (*)$$

Put $\tau=0$ after you derive. I think.

$$= \frac{d}{dz^k} g_{\alpha\beta}(z)\Big|_{z=f_\tau(x)} \frac{df_\tau^k(x)}{d\tau} \frac{\partial f_\tau^\alpha(x)}{\partial x^\mu} \frac{\partial f_\tau^\beta(x)}{\partial x^\nu}\Big|_{\tau=0}$$

$$+ g_{\alpha\beta}(f_\tau(x)) \frac{\partial}{\partial x^\mu}\left(\frac{df_\tau^\alpha(x)}{d\tau}\right) \frac{\partial f_\tau^\beta(x)}{\partial x^\nu}\Big|_{\tau=0}$$

$$+ g_{\alpha\beta}(f_\tau(x)) \frac{\partial f_\tau^\alpha(x)}{\partial x^\mu} \frac{\partial}{\partial x^\nu}\left(\frac{df_\tau^\beta(x)}{d\tau}\right)\Big|_{\tau=0}$$

~~need to be careful. y^μ at $t=0 = x^\mu$ is diff from the $\frac{\partial}{\partial x^\mu}$~~

If we derive w/ respect to x, irrelevant if derive w.r.t. τ and then $\tau=0$ or first $\tau=0$ and then derive? And $f_\tau^k(x) = f_\tau^k(f_\tau^{-1}(y))$?

We now use $\frac{\partial f_\tau^\alpha(x)}{\partial x^\beta}\Big|_{\tau=0} = \frac{\partial x^\alpha}{\partial x^\beta} = \delta^\alpha_\beta$

and $\frac{df_\tau^\alpha(x)}{d\tau} = X^\alpha(y)$

$$= \left(\frac{\partial}{\partial x^k} g_{\alpha\beta}(x)\right) X^k(y(x,0))\,\delta^\alpha_\mu\,\delta^\beta_\nu + g_{\alpha\beta}(x)\frac{\partial}{\partial x^\mu}\left(X^\alpha(y(x,0))\right)\delta^\beta_\nu$$

$$+\, g_{\alpha\beta}(x)\,\delta^\alpha_\mu\,\frac{\partial}{\partial x^\nu}\left(X^\beta(y(x,0))\right)$$

$$= X^k(x)\left(\frac{\partial}{\partial x^k} g_{\mu\nu}(x)\right) + g_{\alpha\nu}(x)\frac{\partial}{\partial x^\mu}X^\alpha(x) + g_{\mu\beta}(x)\frac{\partial}{\partial x^\nu}X^\beta(x)$$

$= (*)$ on the previous page

c) $S_{gr}(g) = -\frac{1}{16\pi G} \int d^4x \sqrt{|g|}\, R(g)$

with $R(g) = g^{\beta\nu} R_{\beta\nu}$, $R_{\beta\nu} = R^{\alpha}_{\beta\alpha\nu}$

$G^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} R_{\alpha\beta} - \frac{1}{2} g^{\mu\nu} R$

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?

Second equality here not even needed? But as $R(\varphi^* g) = \varphi^* R(g)$, this shows that S_{gr} inv. under $(f_t^* g)$? Thus $\frac{d}{dt} S_{gr} = 0$ should hold?

↓ not sure what you ask. :(

$$S_{gr}(f_t^* g) = -\frac{1}{16\pi G} \left\{ \int d^4x \sqrt{|f_t^* g|}\, R(f_t^* g) \right\}$$

$$\left(= -\frac{1}{16\pi G} \left\{ \int d^4x \sqrt{|g(f_t(x))|}\, |D f_t(x)|\, R(f_t^* g) \right\} \right)$$

$$\leadsto \frac{d}{dt} S_{gr}(f_t^* g)\Big|_{t=0} = -\frac{1}{16\pi G} \left\{ \int d^4x \frac{d}{dt}\left(\sqrt{|f_t^* g|}\, R(f_t^* g)\right) + \int d^4x \sqrt{|f_t^* g|}\, \frac{d}{dt}\left(R(f_t^* g)\right) \right\}\Big|_{t=0}$$

$$= -\frac{1}{16\pi G} \left\{ \int d^4x \frac{1}{2\sqrt{|f_t^* g|}} \frac{d}{dt}\left(|f_t^* g|\right) R(f_t^* g) + \int d^4x \sqrt{|f_t^* g|}\, \frac{d}{dt}\left((f_t^* g)^{\beta\nu} R_{\beta\nu}(f_t^* g)\right) \right\}\Big|_{t=0}$$

$$\overset{H3\,b)c)}{=} -\frac{1}{16\pi G} \left\{ \int d^4x \frac{1}{2\sqrt{|f_t^* g|}} |f_t^* g|\, Tr\left[(f_t^* g)^{-1} \frac{d}{dt}(f_t^* g)\right] R(f_t^* g) + \int d^4x \sqrt{|f_t^* g|}\, \frac{d}{dt}\left((f_t^* g)^{\beta\nu} R_{\beta\nu}(f_t^* g)\right) \right\}\Big|_{t=0}$$

$$\overset{R(\varphi^* g) = \varphi^* R(g)}{=} -\frac{1}{16\pi G} \left\{ \int d^4x \frac{\sqrt{|g|}}{2} g^{\mu k} (X_{k;\mu} + X_{\mu;k}) R(g) + \int d^4x \sqrt{|g|}\, R_{\beta\nu}(g) \frac{d}{dt}(f_t^* g)^{\beta\nu}\Big|_{t=0} \right\}$$

$$- \frac{1}{16\pi G} \int d^4x \sqrt{|g|}\, g^{\beta\nu} \frac{d}{dt} f_t^* R_{\beta\nu}(g)\Big|_{t=0}$$

$$(f_t^* g)^{\alpha\beta} (f_t^* g)_{\beta\gamma} = \delta^\alpha_\gamma \overset{\frac{d}{dt}|_{t=0}}{\Longleftrightarrow} 0 = \frac{d}{dt}(f_t^* g)^{\alpha\beta} g_{\beta\gamma} + g^{\alpha\beta} \frac{d}{dt}(f_t^* g)_{\beta\gamma}$$

$$\leadsto \frac{d}{dt}(f_t^* g)^{\alpha\beta} g_{\beta\gamma} = -g^{\alpha\beta} \frac{d}{dt}(f_t^* g)_{\beta\gamma}$$

$$\overset{\cdot g^{\gamma\delta}}{\leadsto} \frac{d}{dt}(f_t^* g)^{\alpha\delta} = -g^{\alpha\beta} g^{\gamma\delta} \frac{d}{dt}(f_t^* g)_{\beta\gamma}$$

$$= -\frac{1}{16\pi G} \left\{ \int d^4x \frac{\sqrt{|g|}}{2} g^{\mu\nu} (X_{\mu;\nu} + X_{\nu;\mu}) R(g) - \int d^4x \sqrt{|g|}\, R_{\beta\nu}(g)\, g^{\beta\mu} g^{\nu k} \frac{d}{dt}(f_t^* g)_{\mu k}\Big|_{t=0} \right\}$$

$$- \frac{1}{16\pi G} \int d^4x \sqrt{|g|}\, g^{\beta\nu} \frac{d}{dt} f_t^* R_{\beta\nu}(g)\Big|_{t=0}$$

$$\overset{\text{remaining}}{=} \frac{1}{16\pi G}\int d^4x \sqrt{|g|}\Big(R_{\alpha\beta}(g)\, g^{\mu\alpha} g^{\nu\beta}\,(X_{\mu;\nu}+X_{\nu;\mu}) - \frac{1}{2} g^{\mu\nu} R(g)\,(X_{\mu;\nu}+X_{\nu;\mu})\Big)$$

$$- \frac{1}{16\pi G}\int d^4x \sqrt{|g|}\; g^{\beta\nu} \frac{d}{d\tau} f_\tau^* R_{\beta\nu}(g)\Big|_{\tau=0}$$

✓

Why should this additional term vanish? → last sheet:

$$= \frac{1}{16\pi G}\int d^4x \sqrt{|g|}\; G^{\mu\nu}(X_{\mu;\nu}+X_{\nu;\mu})$$

$$(f_\tau^* g)_{\mu\nu} = g_{\mu\nu} + \tau(\delta g)_{\mu\nu} + O(\tau^2)$$

$$\frac{d}{d\tau}(f_\tau^* g)_{\mu\nu}\Big|_{\tau=0}$$

$$- \frac{1}{16\pi G}\int d^4x \sqrt{|g|}\; g^{\beta\nu} \frac{d}{d\tau} f_\tau^* R_{\beta\nu}(g)\Big|_{\tau=0}$$

$$S(f_\tau^* g) = S(g) + \tau\, \delta S(g) + O(\tau^2)$$

$$\frac{d}{d\tau}$$

$\overset{?}{}$ Is $S(f_\tau^* g) = f_\tau^* S(g)$?

vanishes $\because$ $\int d^4x \sqrt{|g|}\,\delta R = 0$ we showed before.

d) $0 \overset{!}{=} \frac{d}{dt} S_{gr}(f_t^* g)\big|_{t=0} = \frac{1}{16\pi G}\int d^4x \sqrt{|g|}\, G^{\mu\nu}(X_{\mu;\nu} + X_{\nu;\mu})$

EL Eq 4 (Euler-Lagrange)

$= \frac{2}{16\pi G}\int d^4x \sqrt{|g|}\, G^{\mu\nu} X_{\mu;\nu}$ as $G^{\mu\nu}$ symmetric

To see $G^{\mu\nu} = G^{\nu\mu}$, look at $R_{\beta\nu}$ first:

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$R_{\beta\nu} = R^{\alpha}_{\ \beta\alpha\nu} = g^{\alpha k} R_{k\beta\alpha\nu} = g^{\alpha k}(-R_{k\nu\beta\alpha} - R_{k\alpha\nu\beta})$

Using $R_{k\beta\alpha\nu} + R_{k\nu\beta\alpha} + R_{k\alpha\nu\beta} = 0$ Bianchi Identity

$= g^{\alpha k} R_{k\nu\alpha\beta} - \cancel{g^{\alpha k} R_{k\alpha\nu\beta}}$ ← using sym. of $R_{k\alpha\nu\beta}$ antisym. in $\alpha \leftrightarrow k$

sym. in $\alpha \leftrightarrow k$

$= R_{\nu\beta}$

$\rightsquigarrow G^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} R_{\alpha\beta} - \frac{1}{2} g^{\mu\nu} R$

$= g^{\mu\alpha} g^{\nu\beta} R_{\beta\alpha} - \frac{1}{2} g^{\nu\mu} R$

$\overset{\text{rename}}{=} g^{\nu\alpha} g^{\mu\beta} R_{\alpha\beta} - \frac{1}{2} g^{\nu\mu} R = G^{\nu\mu}$

$= \frac{2}{16\pi G}\int d^4x \sqrt{|g|}\left\{\nabla_\nu (G^{\mu\nu} X_\mu) - G^{\mu\nu}_{\ \ ;\nu} X_\mu\right\}$

But $G^{\mu\nu} X_\mu$ not a skew sym. tensor? Only one comp.?

$\overset{\#3d)}{\underset{G^{\mu\nu}X_\mu = T^\nu}{=}} \frac{2}{16\pi G}\int d^4x \sqrt{|g|}\left\{\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^\nu}(\sqrt{|g|}\, T^\nu) - G^{\mu\nu}_{\ \ ;\nu} X_\mu\right\}$

Yes. not needed to be skew if only one component

$= -\frac{2}{16\pi G}\int d^4x \sqrt{|g|}\, G^{\mu\nu}_{\ \ ;\nu} X_\mu$ if $T^\nu = G^{\mu\nu} X_\mu$ compact support

$\rightsquigarrow G^{\mu\nu}_{\ \ ;\nu} = 0$ as holds $\forall X_\mu$ w/ compact support

From the Einstein eq. $8\pi G T^{\mu\nu} = G^{\mu\nu}$ it then follows that

$T^{\mu\nu}_{\ \ ;\nu} \sim G^{\mu\nu}_{\ \ ;\nu}$ and thus $T^{\mu\nu}_{\ \ ;\nu} = 0$

X doesn't need compact support if $\nabla G = 0$.