Disclaimer

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https://www.physics-and-stuff.com/

I raise no claim to correctness and completeness of the given solutions! This equally applies to the corrections mentioned above.

General Relativity 5. Exercise Marvin Zanke

14.05.2018 H8) $x^0 = t,\ x^1 = r,\ x^2 = \vartheta,\ x^3 = \phi$ sph. coordinates 9/9

non-vanishing components of the metrical tensor

But g_{33} is not sph. symmetric at all?

$g_{00} = e^{\nu(r)},\ g_{11} = -e^{\lambda(r)},\ g_{22} = -r^2,\ g_{33} = -r^2\sin^2\vartheta$ Good

a) use $f'(r) = \frac{\partial}{\partial r} f(r)$ and $g_{\mu\nu,\lambda} := \frac{\partial}{\partial x^\lambda} g_{\mu\nu}$

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- It's obvious, that only $g_{\mu\nu,\lambda} \neq 0$ for $\mu = \nu$, as those are the only non-vanishing components of the metric

what does it mean physically that metric has no t-dep. at all?

- Also, g_{00}, g_{11}, g_{22} only depend on r and g_{33} depends on r and ϑ

$\Rightarrow$ we then only have the non-vanishing partial derivatives

↓ static.

$g_{00,1}, g_{11,1}, g_{22,1}, g_{33,1}$ and $g_{33,2}$, given by:

$$g_{00,1} = \frac{\partial}{\partial r} g_{00} = \nu'(r) e^{\nu(r)}$$
$$g_{11,1} = \frac{\partial}{\partial r} g_{11} = -\lambda'(r) e^{\lambda(r)}$$
$$g_{22,1} = \frac{\partial}{\partial r} g_{22} = -2r$$
$$g_{33,1} = \frac{\partial}{\partial r} g_{33} = -2r\sin^2\vartheta \quad \text{and} \quad g_{33,2} = \frac{\partial}{\partial \vartheta} g_{33} = -2r^2 \sin\vartheta\cos\vartheta$$

b) the Christoffel symbols are defined by $\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\lambda}(g_{\lambda\alpha,\beta} + g_{\lambda\beta,\alpha} - g_{\alpha\beta,\lambda})$

g^{ij} given by what in general? Even if metric not that simple given by $(g_{ij})^{-1}$? do we have to take summation into account then?

First, denote that $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$, i.e. $g_{\mu\nu} g^{\nu\lambda} = \delta^\lambda_\mu$, $g^{\mu\nu} = (g_{\mu\nu})^{-1}$ here (diag.)

We then consider the different cases for $\mu = 0,1,2,3$ in $\Gamma^\mu_{\alpha\beta}$ and find the values for α, β s.t. $\Gamma^\mu_{\alpha\beta} \neq 0$.

all other comp. of g vanish.

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- let $\mu = 0$: $\frac{1}{2} g^{0\lambda}(g_{\lambda\alpha,\beta} + g_{\lambda\beta,\alpha} - g_{\alpha\beta,\lambda})$

$g^{0\lambda} = (g_{0\lambda})^{-1} \neq 0$ only for $\lambda = 0$ $\quad = \frac{1}{2} g^{00}(g_{0\alpha,\beta} + g_{0\beta,\alpha}) \Rightarrow (\alpha,\beta) \in \{(0,1), (1,0)\}$

- let $\mu = 1$: $\frac{1}{2} g^{11}(g_{1\alpha,\beta} + g_{1\beta,\alpha} - g_{\alpha\beta,1}) \Rightarrow (\alpha,\beta) \in \{(1,1), (0,0), (2,2), (3,3)\}$
- let $\mu = 2$: $\frac{1}{2} g^{22}(g_{2\alpha,\beta} + g_{2\beta,\alpha} - g_{\alpha\beta,2}) \Rightarrow (\alpha,\beta) \in \{(1,2), (2,1), (3,3)\}$
- let $\mu = 3$: $\frac{1}{2} g^{33}(g_{3\alpha,\beta} + g_{3\beta,\alpha} - \cancel{g_{\alpha\beta,3}}) \Rightarrow (\alpha,\beta) \in \{(3,1), (1,3), (3,2), (2,3)\}$

then, looking at those equations for fixed μ, α, β, one explicitly finds,

$$\Gamma^0_{01} \underset{\uparrow \atop \Gamma \text{ sym.}}{=} \Gamma^0_{10} = \frac{1}{2} g^{00}\left(\cancel{g_{01,0}} + g_{00,1}\right) = \frac{1}{2} e^{-\nu(r)}\left(\nu'(r) e^{\nu(r)}\right) = \frac{1}{2}\nu'(r)$$

From now on, we will suppress the argument of $\nu(r)$ and $\lambda(r)$

$$\Gamma^1_{00} = \frac{1}{2} g^{11}\left(\cancel{g_{10,0}} + \cancel{g_{10,0}} - g_{00,1}\right) = -\frac{1}{2}\left(-e^{-\lambda}\right)\left(\nu' e^{\nu}\right) = \frac{1}{2}\nu' e^{\nu-\lambda}$$

$$\Gamma^1_{11} = \frac{1}{2} g^{11}\left(g_{11,1} + \cancel{g_{11,1}} - \cancel{g_{11,1}}\right) = \frac{1}{2}\left(-e^{-\lambda}\right)\left(-\lambda' e^{\lambda}\right) = \frac{\lambda'}{2}$$

$$\Gamma^1_{22} = \frac{1}{2} g^{11}\left(\cancel{g_{12,2}} + \cancel{g_{12,2}} - g_{22,1}\right) = -\frac{1}{2}\left(-e^{-\lambda}\right)(-2r) = -r e^{-\lambda}$$

$$\Gamma^1_{33} = \frac{1}{2} g^{11}\left(\cancel{g_{13,3}} + \cancel{g_{13,3}} - g_{33,1}\right) = -\frac{1}{2}\left(-e^{-\lambda}\right)\left(-2r\sin^2\theta\right) = -r\sin^2\theta\, e^{-\lambda}$$

$$\Gamma^2_{21} = \Gamma^2_{12} = \frac{1}{2} g^{22}\left(\cancel{g_{21,2}} + g_{22,1} - \cancel{g_{12,2}}\right) = \frac{1}{2}\left(-\frac{1}{r^2}\right)(-2r) = \frac{1}{r}$$

$$\Gamma^2_{33} = \frac{1}{2} g^{22}\left(\cancel{g_{23,3}} + \cancel{g_{23,3}} - g_{33,2}\right) = -\frac{1}{2}\left(-\frac{1}{r^2}\right)\left(-2r^2\sin\theta\cos\theta\right)$$

$$= -\sin\theta\cos\theta$$

$$\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{2} g^{33}\left(\cancel{g_{31,3}} + g_{33,1}\right) = \frac{1}{2}\left(-\frac{1}{r^2\sin^2\theta}\right)\left(-2r\sin^2\theta\right) = \frac{1}{r}$$

$$\Gamma^3_{23} = \Gamma^3_{32} = \frac{1}{2} g^{33}\left(g_{32,3} + g_{33,2}\right) = \frac{1}{2}\left(-\frac{1}{r^2\sin^2\theta}\right)\left(-2r^2\sin\theta\cos\theta\right) = \frac{\cos\theta}{\sin\theta}$$

$$= \cot\theta$$

c) We now want to find the non-vanishing components of the Ricci-tensor, which is defined by

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$$R^{\beta}{}_{\nu} = g^{\beta k} R_{k\nu} = g^{\beta k} R^{\alpha}{}_{k\alpha\nu} = g^{\beta k} g^{\alpha\lambda} R_{\lambda k\alpha\nu}$$

This definition for $R^{\beta}{}_{\nu}$ correct? Actually only defined $R_{\beta\nu}$ in lecture?. And why should we pull this index up instead of $R_{\beta\nu}$? ...mps, to contract over it though and just use g to pull up

It's correct, for calculating R.

where $$R^{\alpha}{}_{\beta\mu\nu} = \frac{\partial}{\partial x^{\mu}} \Gamma^{\alpha}_{\beta\nu} - \frac{\partial}{\partial x^{\nu}} \Gamma^{\alpha}_{\beta\mu} + \Gamma^{\alpha}_{\lambda\mu} \Gamma^{\lambda}_{\beta\nu} - \Gamma^{\alpha}_{\lambda\nu} \Gamma^{\lambda}_{\beta\mu}$$

We first look at fixed values of β, ν and see which comp. don't vanish.

We will throughout use

$$R^{\alpha}{}_{k\alpha\nu} = \partial_{\alpha} \Gamma^{\alpha}_{k\nu} - \partial_{\nu} \Gamma^{\alpha}_{k\alpha} + \Gamma^{\alpha}_{\lambda\alpha} \Gamma^{\lambda}_{k\nu} - \Gamma^{\alpha}_{\lambda\nu} \Gamma^{\lambda}_{k\alpha}$$

Is $R^{\mu}{}_{\nu}$ sym.? Then it would be enough to show $R^{i}{}_{0} = R^{0}{}_{i} = 0$?!

You can check that using symm. of $R_{\mu\nu\rho\sigma}$. Not sure.

$\bullet$ $$R^{0}{}_{i} = g^{00} R^{\alpha}{}_{0\alpha i}$$

$$= g^{00} \{ \partial_{\alpha} \Gamma^{\alpha}_{0i} - \partial_i \Gamma^{\alpha}_{0\alpha} + \Gamma^{\alpha}_{\lambda\alpha} \Gamma^{\lambda}_{0i} - \Gamma^{\alpha}_{\lambda i} \Gamma^{\lambda}_{0\alpha} \}$$

Γ^{0}_{01} only $\rightsquigarrow$ vanishes, as no t-dep.; all $=0$; Γ^{0}_{01} only $\rightsquigarrow \Gamma^{\alpha}_{0\alpha}\Gamma^{0}_{01}$, all $=0$

$$= g^{00} \{ - \Gamma^{\alpha}_{\lambda i} \Gamma^{\lambda}_{0\alpha} \} = g^{00} \{ - \Gamma^{0}_{1i} \Gamma^{1}_{00} - \Gamma^{1}_{0i} \Gamma^{0}_{01} \} = 0$$

Γ^{0}_{01} and Γ^{1}_{00} only; $=0 \ \forall i$; $=0 \ \forall i$

How to write g^{ii} down properly here if we don't want to sum?

mention it. no standard way.

$\bullet$ $$R^{i}{}_{0} \overset{\text{no sum!}}{=} g^{ii} R^{\alpha}{}_{i\alpha 0}$$

$$= g^{ii} \{ \partial_{\alpha} \Gamma^{\alpha}_{i0} - \underbrace{\partial_0 \Gamma^{\alpha}_{i\alpha}}_{\text{no } t\text{-dep. in } \Gamma} + \Gamma^{\alpha}_{\lambda\alpha} \Gamma^{\lambda}_{i0} - \Gamma^{\alpha}_{\lambda 0} \Gamma^{\lambda}_{i\alpha} \}$$

Γ^{0}_{10} only $\rightsquigarrow$ vanishes as no t-dep.; Γ^{0}_{10} only $\rightsquigarrow \Gamma^{\alpha}_{0\alpha}\Gamma^{0}_{10}$, all $=0$

$$= g^{ii} \{ - \Gamma^{\alpha}_{\lambda 0} \Gamma^{\lambda}_{i\alpha} \} = g^{ii} \{ - \Gamma^{0}_{10} \Gamma^{1}_{i0} - \Gamma^{1}_{00} \Gamma^{0}_{i1} \} = 0$$

Γ^{0}_{10} and Γ^{1}_{00} only; $=0 \ \forall i$; $=0 \ \forall i$

$\bullet$ $$R^{i}{}_{j} \overset{i \neq j?}{=} g^{ii} R^{\alpha}{}_{i\alpha j}$$

$$= g^{ii} \{ \partial_{\alpha} \Gamma^{\alpha}_{ij} - \partial_j \Gamma^{\alpha}_{i\alpha} + \Gamma^{\alpha}_{\lambda\alpha} \Gamma^{\lambda}_{ij} - \Gamma^{\alpha}_{\lambda j} \Gamma^{\lambda}_{i\alpha} \}$$

$\cdot$ $\Gamma^{2}_{12}, \Gamma^{2}_{21}, \Gamma^{3}_{13}, \Gamma^{3}_{31}$ possible but no θ, ϕ dep.

$\Gamma^{3}_{23}, \Gamma^{3}_{32}$ possible, no ϕ dep.

$$= g^{ii}\{ -\partial_j \Gamma^\alpha_{i\alpha} + \Gamma^\alpha_{\lambda\alpha}\Gamma^\lambda_{ij} - \Gamma^\alpha_{\lambda j}\Gamma^\lambda_{i\alpha} \}$$

$\underline{i=1}$: $\Gamma^0_{10}, \Gamma^1_{11}, \Gamma^2_{12}, \Gamma^3_{13}$ $\Rightarrow j \neq 1$ and thus vanishes as only r dependence

$\underline{i=2}$: $\Gamma^3_{23} \Rightarrow j \neq 2$ and thus vanishes, as only θ-dep.

$\underline{i=3}$: $\Gamma^\alpha_{3\alpha} = 0 \ \forall \alpha$

$$= g^{ii}\{ \Gamma^\alpha_{\lambda\alpha}\Gamma^\lambda_{ij} - \Gamma^\alpha_{\lambda j}\Gamma^\lambda_{i\alpha} \}$$

$\Gamma^2_{12}, \Gamma^2_{21}, \Gamma^3_{13}, \Gamma^3_{31}, \Gamma^3_{23}, \Gamma^3_{32}$ possible

$\leadsto \Gamma^\alpha_{3\alpha}$ doesn't yield anything

$\leadsto \Gamma^\alpha_{2\alpha}\Gamma^2_{12}$ or (equal anyway) $\Gamma^\alpha_{2\alpha}\Gamma^2_{21}$ remains

$$= \Gamma^2_{12}\Gamma^3_{23} \quad \text{for } (i,j) \in \{(1,2),(2,1)\}$$

$$= -\Gamma^\lambda_{i0}\Gamma^0_{\lambda j} - \Gamma^\lambda_{ik}\Gamma^k_{\lambda j}$$

Γ^0_{01} or Γ^0_{10}

$\leadsto \Gamma^0_{10}\Gamma^0_{01}$ or $\Gamma^1_{10}\Gamma^0_{10}$ ($k=0$)

but then $i=j$

$$= -\Gamma^\lambda_{ik}\Gamma^k_{\lambda j} = -\cancel{\Gamma^0_{ik}\Gamma^k_{0j}} - \Gamma^l_{ik}\Gamma^k_{lj}$$

$\leadsto \underline{i=1}$: $-\Gamma^1_{11}\cancel{\Gamma^1_{1j}} - \Gamma^2_{12}\cancel{\Gamma^2_{2j}} - \Gamma^3_{13}\Gamma^3_{3j}$ ($j \neq i$)

$\underset{=}{j=2}$ $-\Gamma^3_{13}\Gamma^3_{32}$

$\underline{i=2}$: $-\Gamma^1_{22}\cancel{\Gamma^2_{1j}} - \Gamma^2_{21}\cancel{\Gamma^1_{2j}} - \Gamma^3_{23}\Gamma^3_{3j}$ ($j \neq i$)

$\underset{=}{j=1}$ $-\Gamma^3_{23}\Gamma^3_{31}$

$\underline{i=3}$: $-\Gamma^1_{33}\cancel{\Gamma^3_{1j}} - \Gamma^3_{31}\cancel{\Gamma^1_{3j}} - \Gamma^2_{33}\cancel{\Gamma^3_{2j}} - \Gamma^3_{32}\ldots$ ($j \neq i$)

$\longrightarrow$ those contributions obviously cancel and we thus find, that only

R^β_ν for $\beta = \nu$ (might) be non-vanishing

$$R^0_0 = g^{00} R^\alpha_{0\alpha 0} = g^{00}\{ \partial_\alpha \Gamma^\alpha_{00} - \cancel{\partial_0 \Gamma^\alpha_{0\alpha}} + \Gamma^\alpha_{\lambda\alpha}\Gamma^\lambda_{00} - \Gamma^\alpha_{\lambda 0}\Gamma^\lambda_{0\alpha} \}$$

$$= g^{00}\{ \partial_1 \Gamma^1_{00} + \Gamma^1_{00}\Gamma^\alpha_{1\alpha} - \Gamma^\alpha_{\lambda 0}\Gamma^\lambda_{0\alpha} \}$$

$$= g^{00}\{ \partial_1 \Gamma^1_{00} + \Gamma^1_{00}(\Gamma^0_{10} + \Gamma^1_{11} + \Gamma^2_{12} + \Gamma^3_{13}) - \Gamma^0_{10}\Gamma^1_{00} - \Gamma^1_{00}\Gamma^0_{01} \}$$

$$= e^{-\nu}\{ \frac{\nu''}{2}e^{\nu-\lambda} + \frac{\nu'^2}{2}e^{\nu-\lambda} - \frac{\nu'\lambda'}{2}e^{\nu-\lambda} + \frac{\nu'}{2}e^{\nu-\lambda}(\frac{\nu'}{2} + \frac{2}{r} + \frac{\lambda'}{2}) - (\frac{\nu'}{2})^2 e^{\nu-\lambda} - (\frac{\nu'}{2})^2 e^{\nu-\lambda} \}$$

$$= e^{-\lambda}\{ \frac{\nu''}{2} + \cancel{\frac{\nu'^2}{2}} - \frac{\nu'\lambda'}{2} + (\frac{\nu'}{2})^2 + \frac{\nu'}{r} + \frac{\nu'\lambda'}{4} - \cancel{\frac{\nu'^2}{2}} \}$$

$$= e^{-\lambda}\{ \frac{\nu''}{2} + (\frac{\nu'}{2})^2 - \frac{\lambda'\nu'}{4} + \frac{\nu'}{r} \}$$

$$R^1_1 = g^{11} R^\alpha_{1\alpha 1} = g^{11}\{ \partial_\alpha \Gamma^\alpha_{11} - \partial_1 \Gamma^\alpha_{1\alpha} + \Gamma^\alpha_{\lambda\alpha}\overset{\text{only } \Gamma^1_{11}}{\Gamma^\lambda_{11}} - \Gamma^\alpha_{\lambda 1}\Gamma^\lambda_{1\alpha} \}$$

$$= g^{11}\{ \partial_1 \Gamma^1_{11} - \partial_1(\Gamma^0_{10} + \Gamma^1_{11} + \Gamma^2_{12} + \Gamma^3_{13}) + \Gamma^1_{11}(\Gamma^0_{10} + \Gamma^1_{11} + \Gamma^2_{12} + \Gamma^3_{13})$$
$$- \Gamma^0_{10}\Gamma^0_{01} - \Gamma^1_{11}\Gamma^1_{11} - \Gamma^2_{12}\Gamma^2_{21} - \Gamma^3_{13}\Gamma^3_{31} \}$$

$$= g^{11}\{ \cancel{\frac{\lambda''}{2}} - \frac{\nu''}{2} - \cancel{\frac{\lambda''}{2}} + \cancel{\frac{2}{r^2}} + \frac{\lambda'}{2}(\frac{\nu'}{2} + \frac{\lambda'}{2} + \frac{2}{r}) - (\frac{\nu'}{2})^2 - (\frac{\lambda'}{2})^2 - \cancel{\frac{1}{r^2}} - \cancel{\frac{1}{r^2}} \}$$

$$= -e^{-\lambda}\{ -\frac{\nu''}{2} + \frac{\lambda'\nu'}{4} + \cancel{(\frac{\lambda'}{2})^2} + \frac{\lambda'}{r} - (\frac{\nu'}{2})^2 - \cancel{(\frac{\lambda'}{2})^2} \}$$

$$= e^{-\lambda}\{ \frac{\nu''}{2} + (\frac{\nu'}{2})^2 - \frac{\lambda'\nu'}{4} - \frac{\lambda'}{r} \}$$

$$R^2_2 = g^{22} R^\alpha_{2\alpha 2} = g^{22}\{\partial_\alpha \overset{\Gamma^1_{22}\text{ only}}{\Gamma^\alpha_{22}} - \partial_2 \Gamma^\alpha_{2\alpha} + \Gamma^\alpha_{\lambda\alpha}\overset{\Gamma^1_{22}\text{ only}}{\Gamma^\lambda_{22}} - \Gamma^\alpha_{\lambda 2}\Gamma^\lambda_{2\alpha}\}$$

$$= g^{22}\{\partial_1\Gamma^1_{22} - \partial_2 \Gamma^3_{23} + \Gamma^1_{22}(\Gamma^0_{10}+\Gamma^1_{11}+\Gamma^2_{12}+\Gamma^3_{13})$$

$\cot\theta = \frac{-\sin^2\theta - \cos^2\theta}{\sin^2\theta}$

$$- \Gamma^1_{22}\Gamma^2_{12} - \Gamma^2_{21}\Gamma^1_{22} - \Gamma^3_{23}\Gamma^3_{32}\}$$

$$= -\frac{1}{r^2}\{-e^{-\lambda} + r\lambda' e^{-\lambda} + \frac{1}{\sin^2\theta} - re^{-\lambda}\left(\frac{\nu'}{2}+\frac{\lambda'}{2}+\frac{2}{r}\right)$$

$$+ re^{-\lambda}\frac{1}{r} - \frac{1}{r}(-re^{-\lambda}) - \cot^2\theta\}$$

$$= -\frac{1}{r^2}\{-e^{-\lambda} + r\lambda'e^{-\lambda} + 1 - \frac{r\nu'}{2}e^{-\lambda} - \frac{r\lambda'}{2}e^{-\lambda} \cancel{-2e^{-\lambda}}$$

$$\cancel{+e^{-\lambda}} \cancel{+e^{-\lambda}}\}$$

$$= -\frac{1}{r^2}\{-e^{-\lambda} + \frac{r\lambda'}{2}e^{-\lambda} - \frac{r\nu'}{2}e^{-\lambda} + 1\}$$

$$= e^{-\lambda}\left\{\frac{1}{r^2} + \frac{\nu'-\lambda'}{2r}\right\} - \frac{1}{r^2}$$

$$R^3_3 = g^{33}R^\alpha_{3\alpha3} = g^{33}\{\partial_\alpha\Gamma^\alpha_{33} - \partial_3\overset{=0}{\Gamma^\alpha_{3\alpha}} + \Gamma^\alpha_{\lambda\alpha}\Gamma^\lambda_{33} - \Gamma^\alpha_{\lambda3}\Gamma^\lambda_{3\alpha}\}$$

$$= g^{33}\{\partial_2\Gamma^2_{33} + \partial_1\Gamma^1_{33} + \Gamma^1_{33}\Gamma^\alpha_{1\alpha} + \Gamma^2_{33}\Gamma^\alpha_{2\alpha} - \Gamma^1_{33}\Gamma^3_{13} - \Gamma^3_{31}\Gamma^1_{33}$$

$$- \Gamma^2_{33}\Gamma^3_{23} - \Gamma^3_{32}\Gamma^2_{33}\}$$

$$= -\frac{1}{r^2\sin^2\theta}\{-\cos^2\theta + \sin^2\theta - \sin^2\theta e^{-\lambda} + r\lambda'\sin^2\theta e^{-\lambda}$$

$$- r\sin^2\theta e^{-\lambda}\left(\frac{\nu'}{2}+\frac{\lambda'}{2}+\frac{2}{r}\right) - \cancel{\sin\theta\cos\theta}(\cot\theta)$$

$$+ 2r\sin^2\theta e^{-\lambda}\frac{1}{r} + \cancel{2\sin\theta\cos\theta\cot\theta}\}$$

$$= -\frac{1}{r^2\sin^2\theta}\{\sin^2\theta - \sin^2\theta e^{-\lambda} + r\lambda'\sin^2\theta e^{-\lambda} \cancel{-2\sin^2\theta e^{-\lambda}} - \frac{r\nu'}{2}\sin^2\theta e^{-\lambda}$$

$$- \frac{r\lambda'}{2}\sin^2\theta e^{-\lambda} \cancel{+2\sin^2\theta e^{-\lambda}}\}$$

$$= -\frac{1}{r^2}\{1 - e^{-\lambda} + \frac{r\lambda'}{2}e^{-\lambda} - \frac{r\nu'}{2}e^{-\lambda}\}$$

$$= e^{-\lambda}\left\{\frac{1}{r^2} + \frac{\nu'-\lambda'}{2r}\right\} - \frac{1}{r^2}$$

d) The Ricci scalar is then defined by $R = g^{\beta\nu}R_{\beta\nu} = g^{\beta\nu}g_{\beta\kappa}R^\kappa_\nu = R^\nu_\nu$

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$$\Rightarrow R = e^{-\lambda}\left\{\nu'' + \frac{(\nu')^2}{2} - \frac{\nu'\lambda'}{2} + \frac{\nu'-\lambda'}{r} + \frac{\nu'-\lambda'}{r} + \frac{2}{r^2}\right\} - \frac{2}{r^2}$$

$$= e^{-\lambda}\left\{\nu'' + \frac{(\nu')^2}{2} - \frac{\nu'\lambda'}{2} + \frac{2(\nu'-\lambda')}{r} + \frac{2}{r^2}\right\} - \frac{2}{r^2}$$

Again: In lecture only $G_{\mu\nu}$ or $G^{\mu\nu}$ – which index should be raised/lowered now? Order important?! ⇒ yes, but G symmetric; actually it would ...

e) The Einstein Tensor is defined by $G^\alpha_\beta = g_{\beta\kappa}G^{\alpha\kappa}$ where

$$G^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}R_{\alpha\beta} - \frac{1}{2}g^{\mu\nu}R$$

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$$\Rightarrow G^\alpha_\beta = g_{\beta\kappa}\left(g^{\alpha\lambda}g^{\kappa\rho}R_{\lambda\rho} - \frac{1}{2}g^{\alpha\kappa}R\right) = g^{\alpha\lambda}R_{\lambda\beta} - \frac{1}{2}\delta^\alpha_\beta R$$

$$= R^\alpha_\beta - \frac{1}{2}\delta^\alpha_\beta R$$

This obviously has only diagonal entries!

$$G^0_0 = R^0_0 - \frac{1}{2}R = e^{-\lambda}\left\{\cancel{\frac{\nu''}{2}} + \cancel{\left(\frac{\nu'}{2}\right)^2} - \cancel{\frac{\lambda'\nu'}{4}} + \frac{\nu'}{r} - \cancel{\frac{\nu''}{2}} - \cancel{\frac{(\nu')^2}{4}} + \cancel{\frac{\nu'\lambda'}{4}} - \frac{\nu'-\lambda'}{r} - \frac{1}{r^2}\right\} + \frac{1}{r^2}$$

$$= e^{-\lambda}\left\{\frac{\lambda'}{r} - \frac{1}{r^2}\right\} + \frac{1}{r^2}$$

$$G^1_1 = R^1_1 - \frac{1}{2}R = e^{-\lambda}\left\{\cancel{\frac{\nu''}{2}} + \cancel{\left(\frac{\nu'}{2}\right)^2} - \cancel{\frac{\lambda'\nu'}{4}} - \frac{\lambda'}{r} - \cancel{\frac{\nu''}{2}} - \cancel{\frac{(\nu')^2}{4}} + \cancel{\frac{\nu'\lambda'}{4}} - \frac{\nu'-\lambda'}{r} - \frac{1}{r^2}\right\} + \frac{1}{r^2}$$

$$= e^{-\lambda}\left\{-\frac{\nu'}{r} - \frac{1}{r^2}\right\} + \frac{1}{r^2} = -e^{-\lambda}\left\{\frac{\nu'}{r} + \frac{1}{r^2}\right\} + \frac{1}{r^2}$$

$$G^2_2 = R^2_2 - \frac{1}{2}R = R^3_3 - \frac{1}{2}R = G^3_3$$

$$= e^{-\lambda}\left\{\frac{\nu'-\lambda'}{2r} + \cancel{\frac{1}{r^2}} - \frac{\nu''}{2} - \frac{(\nu')^2}{4} + \frac{\nu'\lambda'}{4} - \frac{\nu'-\lambda'}{r} - \cancel{\frac{1}{r^2}}\right\} + \frac{1}{r^2} - \frac{1}{r^2}$$

$$= e^{-\lambda}\left\{\frac{\lambda'-\nu'}{\cancel{4}_{2}\,r} - \frac{\nu''}{2} - \left(\frac{\nu'}{2}\right)^2 + \frac{\lambda'\nu'}{4\,(4)}\right\}$$

(sorry)