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https://www.physics-and-stuff.com/

I raise no claim to correctness and completeness of the given solutions! This equally applies to the corrections mentioned above.

General Relativity 6. Home exercise

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29.05.2018

8/8 Good

H9) $\psi_{\pm k}(x) = \frac{1}{L^{3/2}} e^{i(k^0x^0 - \vec{k}\vec{x})} \xi_\pm$

$k^0 = \sqrt{|\vec{k}|^2 + m^2}$ $\uparrow \in \mathbb{C}^2$, normalized

a) We use periodic boundary conditions, meaning that for every direction x^1, x^2, x^3, we demand

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$$\psi_{\pm k}(x) = \psi_{\pm k}(x + L_i), \quad L_i = L e_i$$

$$\rightsquigarrow e^{i(k^0x^0 - \vec{k}\cdot\vec{x})} = e^{i(k^0x^0 - \vec{k}\vec{x} - k^iL^i)}$$

$$\rightsquigarrow k^i L = 2\pi n^i \quad \text{(no summation)}$$

$$\rightsquigarrow k^i = \frac{2\pi}{L} n^i \text{ and thus } \vec{k} = \frac{2\pi}{L}\vec{n}$$

b) $\rho(\vec{p}, \vec{x}) = \sum_{k_i} \delta^{(3)}(\vec{p} - \vec{k}_i) |\psi_{\pm k_i}(x)|^2$

$= 2 \sum_{k_i} \delta^{(3)}(\vec{p} - \vec{k}_i) \frac{1}{L^3}$? as $|\psi_{k_i}|^2 \sim |\xi_+|^2 + |\xi_-|^2 = 2$

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Not necessarily a spin $\pm$ available if not all the states filled up to k_F? $\rightsquigarrow$ $\rho(\vec{p}, \vec{x})$ doesn't differ between $\uparrow$ and $\downarrow$; both are possible $\rightsquigarrow$ factor 2

c) In the ground state, i.e. $T=0$, the N fermions occupy all energy levels up to $\epsilon_F = \sqrt{k_F^2 + m^2}$, which yields for the probability

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$$\rho(\vec{p}, \vec{x}) = \sum_{\xi_\pm} \sum_{|\vec{k}| \leq k_F} \delta^3(\vec{p} - \vec{k}) |\psi_{\pm k}(x)|^2 = 2 \sum_{|\vec{k}| \leq k_F} \delta^{(3)}(\vec{p} - \vec{k}) \frac{1}{L^3}$$

Only $\rho(|\vec{p}|, \vec{x})$ possible or w/o modulus okay? $\rightsquigarrow$ in (b) each component equal; in (c) we have rot. sym.

d) the volume of an element in momentum space is given by $\Delta k = \frac{2\pi}{L} \rightsquigarrow (\Delta k)^3 = \left(\frac{2\pi}{L}\right)^3$ which becomes very small for large L (or equivalently $V = L^3$). Thus, the summation elements come closer and closer to d^3k ", i.e. a continuous distribution

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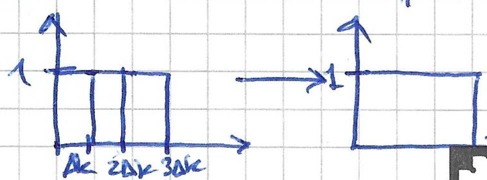

and thus

$$\left(\frac{2\pi}{L}\right)^3 \sum_{|\vec{k}| \leq k_F} \longrightarrow \int_{|\vec{k}| \leq k_F} d^3k$$

e) We had $\rho(\vec{p},\vec{x}) = 2 \sum_{|\vec{k}|<k_F} \delta^{(3)}(\vec{p}-\vec{k}) \frac{1}{L^3}$, which turns into

$$\rho(\vec{p},\vec{x}) = \frac{2}{(2\pi)^3} \int_{|\vec{k}|<k_F} \delta^{(3)}(\vec{p}-\vec{k}) d^3k = \frac{2}{(2\pi)^3} \int_{\vec{k}} d^3k\, \theta(k_F - |\vec{k}|)\, \delta^{(3)}(\vec{p}-\vec{k})$$

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$$= \frac{2}{(2\pi)^3} \theta(k_F - |\vec{p}|)$$

↑ correct that θ-distr. stays? $\neq \frac{2}{(2\pi)^3}$ if ordin[ary] δ-distrn. after first step?
$\rightsquigarrow$ In fact evaluated $\int_{|\vec{k}|<k_F} d^3k\, \delta(\vec{p}-\vec{k})$ for 2 cases $|p|>k_F>|p|$

f) We calculate $n = \frac{N}{V} = \int d^3k\, \rho(\vec{k},\vec{x}) = \frac{2}{(2\pi)^3} 4\pi \int_0^{k_F} dk\, k^2$

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$$= \frac{1}{\pi^2} \frac{1}{3} k^3 \Big|_0^{k_F} = \frac{1}{3\pi^2} k_F^3$$

$$\rightsquigarrow k_F = (3\pi^2 n)^{1/3}$$

Why is $n = \int d^3k\, \rho(\vec{k},\vec{x})$ still "per" $\vec{x}$? I.e. to find a particle at $\vec{x}$?
$\rightsquigarrow$ independen[t] of $\vec{x}$ and thus it's the density to have particle w/ an[y] mom. k

The energy density is given by

$$\epsilon = \frac{E}{V} = \frac{2}{(2\pi)^3} 4\pi \int_0^{k_F} dk\, k^2 \sqrt{m^2+k^2} =$$

done here for ϵ

$$= \frac{1}{8\pi^2} \left\{ k\sqrt{m^2+k^2}\,(m^2+2k^2) - m^4 \log\left(\sqrt{m^2+k^2}+k\right) \right\} \Big|_0^{k_F}$$

$$= \frac{1}{8\pi^2} \left\{ k_F\sqrt{m^2+k_F^2}\,(m^2+2k_F^2) - m^4 \log\left(\sqrt{m^2+k_F^2}+k_F\right) + m^4 \log(m) \right\}$$

What for did we do all those substitutions $x = \frac{k_F}{m}$, $y = \frac{k}{m}$ etc. in the lecture to calculate the integral $\int dk\, k^2\sqrt{m^2+k^2}$?

The pressure is given by $p = -\frac{\partial E}{\partial V}$, $E = \epsilon \cdot V$

$$\rightsquigarrow p = -\epsilon - V \frac{\partial \epsilon}{\partial k_F} \frac{\partial k_F}{\partial V}$$

$$\frac{\partial}{\partial V} k_F = \frac{\partial}{\partial V} \left(3\pi^2 \frac{N}{V}\right)^{1/3} = (3\pi^2 N)^{1/3} \left(-\tfrac{1}{3}\right) V^{-4/3} = -\frac{1}{3} \frac{k_F}{V}$$

$$\frac{\partial}{\partial k_F} \epsilon = \frac{1}{\pi^2} k_F^2 \sqrt{m^2+k_F^2}$$

Should we really calculate p here? or only for the cases in g)?
$\rightsquigarrow$ See below

$$\rightsquigarrow p = -\epsilon + \frac{1}{3} k_F \frac{1}{\pi^2} k_F^2 \sqrt{m^2+k_F^2} = -\epsilon + \frac{k_F^3}{3\pi^2} \sqrt{m^2+k_F^2}$$

not sure about this

$$P_{rm} = \frac{1}{3\pi^2} \int_0^{k_F} dp\, p^4 \left(p^2+m^2\right)^{-1/2}$$

$$P_m = \frac{1}{3} \int d^3p\, \rho \frac{p^2}{\sqrt{p^2+m^2}}$$ Ansatz

g) For $m \to 0$, we get

$$\epsilon = \frac{1}{\pi^2}\int_0^{k_F} dk\, k^3 = \frac{1}{4\pi^2} k_F^4$$

$$\rightsquigarrow p = -\frac{1}{4\pi^2} k_F^4 + \frac{k_F^4}{3\pi^2} = \frac{k_F^4}{12\pi^2} = \frac{1}{3}\epsilon$$

For $m \to \infty$, we get

$$\epsilon = \frac{1}{\pi^2} m \int_0^{k_F} dk\, k^2 = \frac{m}{3\pi^2} k_F^3$$

$$\rightsquigarrow p = -\frac{m}{3\pi^2} k_F^3 + \frac{k_F^3}{3\pi^2} m = 0$$

h) System of neutrons, protons, electrons w/ fermi momenta k_F^n, k_F^p, k_F^e

$$\rightsquigarrow n^n = \frac{1}{3\pi^2} k_F^{n\,3}$$

$$n^p = \frac{1}{3\pi^2} k_F^{p\,3}$$

$$n^e = \frac{1}{3\pi^2} k_F^{e\,3}$$

For a vanishing charge of the total system, we need $N^e = N^p$ and thus $n^e = n^p \Leftrightarrow k_F^p = k_F^e$.