Disclaimer

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I raise no claim to correctness and completeness of the given solutions! This equally applies to the corrections mentioned above.

General Relativity 7. Home exercise — Marvin Zanke

7/7 Good

(2)

05.06.2018 H10) $\epsilon(r) = \begin{cases} \epsilon_0 = \text{const}, & r < r_0 \\ 0, & r > r_0 \end{cases}$

$$g_{00} = e^{\nu(r)},\ g_{11} = -e^{\lambda(r)},\ g_{22} = -r^2,\ g_{33} = -r^2\sin^2\theta$$

Radius r_0 free input parameter – What else?

meaning?

$\rightsquigarrow$ If we had ϵ and p we could compute r_0

Can choose j^μ arbitrary s.t. $g_{\mu\nu}j^\mu j^\nu$ fulfilled?

$$G^0_0 = \frac{1}{r^2} - \left(\frac{1}{r^2} - \frac{\lambda'(r)}{r}\right)e^{-\lambda(r)}$$

$$G^1_1 = \frac{1}{r^2} - \left(\frac{1}{r^2} + \frac{\nu'(r)}{r}\right)e^{-\lambda(r)}$$

$$G^2_2 = G^3_3 = -\frac{1}{2}\left(\nu''(r) + \frac{\nu'^2(r)}{2} - \frac{\lambda'(r)\nu'(r)}{2} + \frac{\nu' - \lambda'}{r}\right)e^{-\lambda(r)}$$

$$T^0_0 = \epsilon,\quad T^1_1 = -p,\quad T^2_2 = -p,\quad T^3_3 = -p,\quad G^\mu_\nu = 8\pi G T^\mu_\nu$$

a)

1/G

$$G^0_0 = 8\pi G T^0_0 \iff 8\pi G\epsilon_0 = \frac{1}{r^2} - \left(\frac{1}{r^2} - \frac{\lambda'}{r}\right)e^{-\lambda}$$

$T^{\alpha\beta} = (\epsilon+p)j^\alpha j^\beta - p g^{\alpha\beta}$

$g_{\mu\nu}j^\mu j^\nu = 1$?

$\rightsquigarrow T^{00} = \frac{1}{g_{00}}(\epsilon+p) - p g^{00}$

$\rightsquigarrow T^0_0 = (\epsilon+p) - p$

also if g not diagonal?

I would think so.

$\rightsquigarrow$ more difficult if $T_{\mu\nu}$/$g_{\mu\nu}$ not diagonal.

$$\iff 8\pi G\epsilon_0 r^2 = 1 - (1 - \lambda' r)e^{-\lambda} = 1 - e^{-\lambda} + \lambda' r e^{-\lambda}$$

$$= \frac{d}{dr}\left(r - re^{-\lambda}\right)$$

$$\iff 8\pi G \underbrace{\int_0^r d\rho\, \rho^2 \epsilon_0}_{= \frac{M(r)}{4\pi}} = r - re^{-\lambda} = r\left(1 - e^{-\lambda}\right)$$

$$\iff 2GM(r) = r\left(1 - e^{-\lambda}\right)$$

$$\iff e^{-\lambda} = 1 - \frac{2GM(r)}{r}$$

$$\iff e^{\lambda} = \left(1 - \frac{2GM(r)}{r}\right)^{-1}$$

$\underline{r < r_0}$: $M(r) = 4\pi \int_0^r d\rho\, \rho^2 \epsilon_0 = \frac{4\pi}{3} r^3 \epsilon_0$

$$\rightsquigarrow e^{\lambda} = \left(1 - \frac{8\pi G r^3 \epsilon_0}{3r}\right)^{-1} = \left(1 - \frac{r^2}{R^2}\right)^{-1}$$

$$R^2 = \frac{3}{8\pi G\epsilon_0}$$

Why not evaluate M_0 as well? $\rightsquigarrow$ can be easily cal.

$\rightarrow 1 - \frac{1}{r}\left(\frac{r_0^3}{R^2}\right)$

$\underline{r > r_0}$: $M(r) = 4\pi \int_0^{r_0} d\rho\, \rho^2 \epsilon_0 \equiv M_0 = \frac{4\pi r_0^3 \epsilon_0}{3}$

$$\rightsquigarrow e^{\lambda} = \left(1 - \frac{2GM_0}{r}\right)^{-1}$$

b) Also had a third eq. arising from $T^{\mu\nu}{}_{;\nu}$ or the Einstein eq.:

$$-2p'(r) = \nu'(r)(p(r) + \epsilon(r))$$

$$\Leftrightarrow p'(r) = -\frac{\nu'(r)}{2}(p(r)+\epsilon_0) = -\frac{\nu'(r)}{2}p(r) - \frac{\nu'(r)}{2}\epsilon_0 \quad (*)$$

$\rightsquigarrow$ first solve hom. diff. eq. $p'(r) = -\frac{\nu'(r)}{2}p(r)$

In full: $\nu' = \frac{-2p'}{\epsilon_0+p} = -2\log(\epsilon_0+p)'$

$$\rightsquigarrow p(r) = Ce^{-\nu/2}$$

$\rightsquigarrow$ Then use the special solution $p(r) = -\epsilon_0$ for the inhom. diff eq. and add it to $p(r)$.

$$\rightsquigarrow p(r) = Ce^{-\nu/2} - \epsilon_0 \quad \text{solves } (*)$$

$$\Leftrightarrow p(r) + \epsilon_0 = Ce^{-\nu/2}$$

c) From $G^0{}_0 = 8\pi G T^0{}_0$ and $G^1{}_1 = 8\pi G T^1{}_1$, we find

$$8\pi G\epsilon_0 = \frac{1}{r^2} - \left(\frac{1}{r^2} - \frac{\lambda'}{r}\right)e^{-\lambda}$$

$$-8\pi G p = \frac{1}{r^2} - \left(\frac{1}{r^2} + \frac{\nu'}{r}\right)e^{-\lambda}$$

$$\rightsquigarrow 8\pi G(p+\epsilon_0) = \frac{\lambda'+\nu'}{r}e^{-\lambda} = 8\pi G ce^{-\nu/2}$$

$$\rightsquigarrow (\lambda'+\nu')e^{\nu/2} = 8\pi G cre^{\lambda}$$

$$\rightsquigarrow \nu'e^{\nu/2} = 8\pi G cre^{\lambda} - \lambda'e^{\nu/2}$$

$$y = e^{\nu/2} \Leftrightarrow 2y' = 8\pi G cre^{\lambda} - \lambda' y$$

Use the ansatz

$$y = e^{\nu/2} = A - B\left(1-\frac{r^2}{R^2}\right)^{1/2} \rightsquigarrow y' = \frac{B\lambda'}{2}e^{-\lambda/2}$$

$$= A - Be^{-\lambda/2}$$

$$\rightsquigarrow \cancel{B\lambda'e^{-\lambda/2}} \overset{!}{=} 8\pi G cre^{\lambda} - A\lambda' + \cancel{B\lambda'e^{-\lambda/2}}$$

$$\Leftrightarrow 8\pi G cre^{\lambda} = A\lambda' \Leftrightarrow c = \frac{A\lambda'}{8\pi G r}e^{-\lambda}$$

$$\left| \; e^{-\lambda} = \left(1 - \frac{r^2}{R^2}\right) \Leftrightarrow -\lambda' e^{-\lambda} = -\frac{2r}{R^2} \Leftrightarrow \lambda'e^{-\lambda} = \frac{2r}{R^2}\right.$$

$$\Leftrightarrow c = \frac{2A}{8\pi G R^2}$$

d) From b) $p(r) = Ce^{-\nu/2} - \epsilon_0$

From a) $R^2 = \dfrac{3}{8\pi G \epsilon_0} \Leftrightarrow \epsilon_0 = \dfrac{3}{8\pi G R^2}$

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Using c), we find

$$p(r) = Ce^{-\nu/2} - \epsilon_0 = \frac{C}{A - B\sqrt{1-\frac{r^2}{R^2}}} - \frac{3}{8\pi G R^2}$$

$$= \frac{8\pi G R^2 C - 3A + 3B\sqrt{1-\frac{r^2}{R^2}}}{8\pi G R^2 \left\{A - B\sqrt{1-\frac{r^2}{R^2}}\right\}}$$

$$\overset{C \text{ from c)}}{=} \frac{3B\sqrt{1-\frac{r^2}{R^2}} - A}{A - B\sqrt{1-\frac{r^2}{R^2}}}$$

e) To fix A and B, we use $p(r_0) = 0$ and $e^{\nu(r_0)} = 1 - \dfrac{2GM_0}{r_0}$

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i.e. continuity of p and e^ν at the boundary of the star, as from the lecture

$$e^{\nu(r)} = \left(1 - \frac{2GM(r_0)}{r}\right) \text{ for } r > r_0$$

$$p(r) = 0 \text{ for } r > r_0$$

$p(r_0) = 0$ as well on the boundary itself?
⇝ we don't know exactly what it's at the boundary microscopically; but macroscopically take $p(r_0+\delta)$ for $\delta \to 0$, which can be assumed to vanish

$$\rightsquigarrow p(r_0) = \frac{1}{8\pi G R^2} \frac{3B\sqrt{1-\frac{r_0^2}{R^2}} - A}{A - B\sqrt{1-\frac{r_0^2}{R^2}}} \overset{!}{=} 0$$

$$\Longleftrightarrow A = 3B\sqrt{1-\frac{r_0^2}{R^2}} = 3Be^{-\lambda(r_0)/2}$$

$$e^{\nu(r_0)/2} = A - B\sqrt{1-\frac{r_0^2}{R^2}} \overset{!}{=} \sqrt{1-\frac{2GM_0}{r_0}}$$

$$= A - A/3 = \frac{2A}{3}$$

$$\Longleftrightarrow A = \frac{3}{2}\sqrt{1-\frac{2GM_0}{r_0}}$$

Can those A, B be simplified further?
⇝ not in tutorial

$$\rightsquigarrow B = \frac{A}{3\sqrt{1-\frac{r_0^2}{R^2}}} = \frac{\sqrt{1-\frac{2GM_0}{r_0}}}{2\sqrt{1-\frac{r_0^2}{R^2}}}$$