Disclaimer

General Relativity 8. Home exercise

Marvin Zanke

11/12

H1)

11.06.2018 Coordinates $y^0 = \bar{t}$, $y^1 = \bar{r}$, $y^2 = \vartheta$, $y^3 = \phi$

with Schwartzschild metric

$$\bar{g}_{00} = \left(1 - \frac{r_g}{\bar{r}}\right), \quad \bar{g}_{11} = -\left(1 - \frac{r_g}{\bar{r}}\right)^{-1}$$

why r_g instead of r_S here?

just notation, no reason.

$$\bar{g}_{22} = -\bar{r}^2, \quad \bar{g}_{33} = -\bar{r}^2 \sin^2\vartheta$$

The clouds comoving coordinates $x^0 = t$, $x^1 = r$, $x^2 = \vartheta$, $x^3 = \phi$ mapped onto Schwarzschild coordinates by

$$\bar{t}(t,r) \equiv \left(\frac{1 - a r_0^2}{a}\right)^{1/2} \int_{S(t,r)}^{1} d\rho \, \frac{\left(\frac{\rho}{1-\rho}\right)^{1/2}}{1 - \frac{a r_0^2}{\rho}}$$

$$\bar{r}(t,r) = r f(t), \quad r_0 = r(t=0)$$

$$S(t,r) = 1 - \left(\frac{1 - ar^2}{1 - a r_0^2}\right)^{1/2} (1 - f(t))$$

a) Let $A(t,r) \equiv \left(1 - \frac{ar^2}{f(t)}\right)^{-1}$

2

$$B(t,r) \equiv \frac{f(t)}{S(t,r)} \left(\frac{1 - ar^2}{1 - a r_0^2}\right)^{1/2} \frac{\left(1 - \frac{a r_0^2}{S(t,r)}\right)^2}{1 - \frac{ar^2}{f(t)}}$$

Now $r = r_0 \Rightarrow S(t, r_0) = f(t)$

$$\Rightarrow A(t, r_0) = \left(1 - \frac{a r_0^2}{f(t)}\right)^{-1} \stackrel{!}{=} \left(1 - \frac{r_g}{\bar{r}(t, r_0)}\right)^{-1}$$

$$= \left(1 - \frac{r_g}{r_0 f(t)}\right)^{-1}$$

why need two conditions? True for each individually? Or need both for "iff and only if" statement?

$$\Leftrightarrow a r_0^2 = \frac{r_g}{r_0}$$

$$\Leftrightarrow a r_0^3 = r_g$$

one is comoving coordinates, AND other is Schwarzschild.

$$B(t, r_0) = \left(1 - \frac{a r_0^2}{f(t)}\right) \stackrel{!}{=} \left(1 - \frac{r_g}{\bar{r}(t, r_0)}\right)$$

we need comoving for studying motion of particles.

$$= \left(1 - \frac{r_g}{r_0 f(t)}\right)$$

$$\Leftrightarrow a r_0^3 = r_g$$

Then $a = \frac{8}{3}\pi G \rho(0)$, $r_g = 2GM$

$\rightsquigarrow$ $r_g = \frac{8}{3}\pi G \rho(0) r_0^3 = 2GM$

with $M = \frac{4}{3}\pi \rho(0) r_0^3$

which is then the mass contained in a sphere with radius r_0

$M = \rho(0) \cdot V$, $V = \frac{4\pi}{3} r_0^3$ ✓

But why is ρ constant "along" $r \to r_0$ i.e. $\rho = \rho(0)$?

→ here ρ is not a function of distance. We assume constant density cloud at beginning

b) we then define $\tilde{g}_{00} = B$, $\tilde{g}_{11} = -A$, $\tilde{g}_{22} = -r^2 f^2(t)$,

2.5

$\tilde{g}_{33} = -r^2 f^2(t) \sin^2\theta$ where the other comp. vanish

We want to prove $g_{\alpha\beta} \overset{!}{=} \tilde{g}_{\mu\nu} \frac{\partial y^\mu}{\partial x^\alpha} \frac{\partial y^\nu}{\partial x^\beta}$ with

$g_{00} = 1$, $g_{11} = \frac{-f^2(t)}{1-ar^2}$, $g_{22} = -f^2(t) r^2$, $g_{33} = -f^2(t) r^2 \sin^2\theta$

the interior dust cloud metric in terms of the comoving coordinates $x^0 = t$, $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$.

From the lecture, we also know the diff. eq. for f:

$$\left(\frac{df}{dt}\right)^2 = a\left(\frac{1}{f(t)} - 1\right)$$

✓ What is the metric $\tilde{g}$ for if g is already the interior dust cloud metric?

$\rightsquigarrow$ $\tilde{g}(r=r_0) \leftrightarrow g$

$\tilde{g}$ goes over to Schw. sch. metric for $r=r_0$ but can also be transformed to be the interior dust cloud metric

$\rightsquigarrow$ $$g_{\alpha\beta} \doteq \tilde{g}_{00} \frac{\partial y^0}{\partial x^\alpha} \frac{\partial y^0}{\partial x^\beta} + \tilde{g}_{11} \frac{\partial y^1}{\partial x^\alpha} \frac{\partial y^1}{\partial x^\beta} + \tilde{g}_{22} \frac{\partial y^2}{\partial x^\alpha} \frac{\partial y^2}{\partial x^\beta} + \tilde{g}_{33} \frac{\partial y^3}{\partial x^\alpha} \frac{\partial y^3}{\partial x^\beta}$$

Calculate: $\frac{\partial y^0}{\partial x^0} = \frac{\partial}{\partial t} \bar{t}(t,r)$, $\frac{\partial y^0}{\partial x^1} = \frac{\partial}{\partial r} \bar{t}(t,r)$

$\frac{\partial y^0}{\partial x^2} = 0$, $\frac{\partial y^0}{\partial x^3} = 0$

$\frac{\partial y^1}{\partial x^0} = \frac{\partial}{\partial t} \bar{r}(t,r)$, $\frac{\partial y^1}{\partial x^1} = \frac{\partial}{\partial r} \bar{r}(t,r)$, $\frac{\partial y^1}{\partial x^2} = 0$, $\frac{\partial y^1}{\partial x^3} = 0$

$\frac{\partial y^2}{\partial x^0} = 0$, $\frac{\partial y^2}{\partial x^1} = 0$, $\frac{\partial y^2}{\partial x^2} = 1$, $\frac{\partial y^2}{\partial x^3} = 0$

$\frac{\partial y^3}{\partial x^0} = 0$, $\frac{\partial y^3}{\partial x^1} = 0$, $\frac{\partial y^3}{\partial x^2} = 0$, $\frac{\partial y^3}{\partial x^3} = 1$

We start with: $g_{00} \overset{!}{=} \tilde{g}_{00} \frac{\partial y^0}{\partial x^0} \frac{\partial y^0}{\partial x^0} + \tilde{g}_{11} \frac{\partial y^1}{\partial x^0} \frac{\partial y^1}{\partial x^0}$

$$\rightsquigarrow B \left\{ \frac{\partial}{\partial t} \bar{t}(t,r) \right\}^2 - A \left\{ \frac{\partial}{\partial t} \bar{r}(t,r) \right\}^2$$

e.g. $\int_0^{f(t,x)} d\varrho\, \varrho = \frac{1}{2} f^2(t,x)$ (not a proof, just an example.)

$$\rightsquigarrow \frac{\partial}{\partial t} \left\{ \frac{1}{2} f^2(t,x) \right\} = f(t,x) \frac{\partial f}{\partial t}$$

$$\frac{\partial}{\partial t} \bar{t}(t,r) = \left(\frac{1-ar_0^2}{a} \right)^{1/2} (-1) \frac{\left(\frac{s(t,r)}{1-s(t,r)} \right)^{1/2}}{1 - \frac{ar_0^2}{s(t,r)}} \frac{\partial s(t,r)}{\partial t}$$

AND

$$\frac{\partial}{\partial t} \bar{r}(t,r) = \left(r \frac{\partial}{\partial t} f(t) \right) \rightsquigarrow \left(\frac{\partial}{\partial t} \bar{r}(t,r) \right)^2 = r^2 a \left(\frac{1}{f(t)} - 1 \right)$$

$$= \frac{1-ar_0^2}{a} \frac{\frac{s(t,r)}{1-s(t,r)}}{\left(1 - \frac{ar_0^2}{s(t,r)}\right)^2} \left(\frac{\partial s}{\partial t} \right)^2 B - A r^2 a \left(\frac{1}{f(t)} - 1 \right)$$

$$\left(\frac{\partial s}{\partial t} \right)^2 = \frac{1-ar^2}{1-ar_0^2} \left(\frac{\partial f}{\partial t} \right)^2 = \underline{\frac{1-ar^2}{1-ar_0^2} a \left(\frac{1}{f(t)} - 1 \right)}$$

had to check this at end.

Can also derive a with f and then show that f has to fulfill the diff. eq.

$$= (1-ar^2) \left(\frac{1}{f(t)} - 1 \right) \frac{\frac{s(t,r)}{1-s(t,r)}}{\cancel{\left(1 - \frac{ar_0^2}{s(t,r)}\right)^2}} \frac{f(t)}{\cancel{s(t,r)}} \underbrace{\left(\frac{1-ar^2}{1-ar_0^2} \right)^{1/2}}_{(*)} \frac{\cancel{\left(1 - \frac{ar_0^2}{s(t,r)}\right)^2}}{1 - \frac{ar^2}{f(t)}}$$

$$- \frac{\frac{r^2 a}{f(t)} - r^2 a}{1 - \frac{ar^2}{f(t)}}$$

$$\frac{1}{1-s(t,r)} = \underbrace{\left(\frac{1-ar_0^2}{1-ar^2} \right)^{1/2}}_{(*)} \frac{1}{1-f(t)}$$

$$= (1-ar^2) \left(\frac{1}{f(t)} - 1 \right) \frac{f(t)}{1-f(t)} \frac{1}{1 - \frac{ar^2}{f(t)}} - \frac{\frac{r^2 a}{f(t)} - r^2 a}{1 - \frac{ar^2}{f(t)}}$$

$$= \frac{1}{1 - \frac{ar^2}{f(t)}} \left\{ (1-ar^2) \frac{1-f(t)}{1-f(t)} - r^2 a \left(\frac{1}{f(t)} - 1 \right) \right\}$$

$$= \frac{1}{1 - \frac{ar^2}{f(t)}} \left\{ 1 - \frac{ar^2}{f(t)} \right\} = 1 = g_{00}$$

Now: $g_{11} \overset{!}{=} \tilde{g}_{00} \frac{\partial y^0}{\partial x^1} \frac{\partial y^0}{\partial x^1} + \tilde{g}_{11} \frac{\partial y^1}{\partial x^1} \frac{\partial y^1}{\partial x^1}$

$$\rightsquigarrow \; B \left(\frac{\partial}{\partial r} \bar{t}(t,r) \right)^2 - A \left(\frac{\partial}{\partial r} \bar{r}(t,r) \right)^2$$

$$\frac{\partial}{\partial r} \bar{t}(t,r) = \left(\frac{1 - a r_0^2}{a} \right)^{1/2} (-1) \frac{\left(\frac{s(t,r)}{1 - s(t,r)} \right)^{1/2}}{1 - \frac{a r_0^2}{s(t,r)}} \left(\frac{\partial s}{\partial r} \right)^2$$

$$\frac{\partial}{\partial r} \bar{r}(t,r) = f(t)$$

$$\frac{\partial s}{\partial r} = -\frac{1}{2} \frac{-2ar}{(1-ar^2)^{1/2} (1 - a r_0^2)^{1/2}} (1 - f(t))$$

$$= \frac{ar(1-f(t))}{(1-ar^2)^{1/2} (1-ar_0^2)^{1/2}}$$

$$= \left(\frac{1-ar_0^2}{1-ar^2} \right)^{1/2} \frac{1}{(1-f(t))}$$

$$= \frac{f(t)}{s\cancel{(t,r)}} \left(\frac{1-ar^2}{1-ar_0^2} \right)^{1/2} \frac{\cancel{\left(1 - \frac{a r_0^2}{s(t,r)}\right)^2}}{1 - \frac{ar^2}{f(t)}} \cancel{\left(\frac{1 - a r_0^2}{a} \right)} \frac{\frac{s(t,r)}{1 - s(t,r)}}{\cancel{\left(1 - \frac{a r_0^2}{s(t,r)}\right)^2}} \frac{a^{\cancel{2}} r^2 (1 - f(t))^2}{(1-ar^2)\cancel{(1-ar_0^2)}}$$

$$- \frac{f(t)^2}{1 - \frac{ar^2}{f(t)}}$$

$$= \frac{f(t)}{1-f(t)} \frac{1}{1 - \frac{ar^2}{f(t)}} \frac{a \cdot r^2 (1-f(t))^2}{1-ar^2} - \frac{f(t)^2}{1 - \frac{ar^2}{f(t)}}$$

$$= \frac{f(t)}{1 - \frac{ar^2}{f(t)}} \left\{ (1-f(t)) \frac{ar^2}{1-ar^2} - f(t) \right\}$$

$$= \frac{f^2(t)}{f(t) - ar^2} \left\{ \frac{ar^2 - ar^2 f(t) - (1-ar^2) f(t)}{1-ar^2} \right\}$$

$$= \frac{f^2(t)}{f(t) - ar^2} \left\{ \frac{ar^2 - f(t)}{1-ar^2} \right\} = -\frac{f^2(t)}{1-ar^2} = g_{11}$$

Also: $g_{22} \overset{!}{=} \tilde{g}_{22} \frac{\partial y^2}{\partial x^2} \frac{\partial y^2}{\partial x^2}$

$\Leftrightarrow -f^2(t)\, r^2 = -r^2 f^2(t)$ ✓

AND $g_{33} \overset{!}{=} \tilde{g}_{33} \frac{\partial y^3}{\partial x^3} \frac{\partial y^3}{\partial x^3}$

$\Leftrightarrow -f^2(t)\, r^2 \sin^2\vartheta = -r^2 f^2(t) \sin^2\vartheta$ ✓

We also need to check g_{0i} and g_{jk} ($j<k$) as g symmetric

$$g_{01} \overset{!}{=} \tilde{g}_{00} \frac{\partial y^0}{\partial x^0} \frac{\partial y^0}{\partial x^1} + \tilde{g}_{11} \frac{\partial y^1}{\partial x^0} \frac{\partial y^1}{\partial x^1} + \tilde{g}_{22} \frac{\partial y^2}{\partial x^0} \frac{\partial y^2}{\partial x^1} + \tilde{g}_{33} \frac{\partial y^3}{\partial x^0} \frac{\partial y^3}{\partial x^1}$$

$$\rightsquigarrow B \left\{ \frac{\partial}{\partial t} \bar{t}(t,r) \right\} \left\{ \frac{\partial}{\partial r} \bar{t}(t,r) \right\} - A \left\{ \frac{\partial}{\partial t} \bar{r}(t,r) \right\} \left\{ \frac{\partial}{\partial r} \bar{r}(t,r) \right\}$$

$$= \frac{f(t)}{\cancel{s(t,r)}} \left(\frac{1-ar^2}{1-ar_0^2} \right)^{1/2} \frac{\cancel{\left(1 - \frac{ar_0^2}{s(t,r)}\right)^2}}{1 - \frac{ar^2}{f(t)}} \left(\frac{\cancel{1-ar_0^2}}{\cancel{a}} \right) \left(\frac{\frac{\cancel{s(t,r)}}{1-s(t,r)}}{\cancel{\left(1 - \frac{ar_0^2}{s(t,r)}\right)^2}} \frac{ar(1-f(t))}{\cancel{(1-ar^2)^{1/2}}\cancel{(1-ar_0^2)^{1/2}}} \cancel{\left(\frac{1-ar^2}{1-ar_0^2}\right)^{1/2}} \left(\frac{1-ar_0^2}{1-ar^2} \right)^{1/2} \frac{1}{1-f(t)} \right)$$

$$- \frac{r \left(\frac{\partial f}{\partial t} \right) f(t)}{1 - \frac{ar^2}{f(t)}}$$

$$= \frac{f(t)}{\cancel{1-f(t)}} \frac{1}{1 - \frac{ar^2}{f(t)}} r \cancel{(1-f(t))} \frac{\partial f}{\partial t} - \frac{r f(t) \frac{\partial f}{\partial t}}{1 - \frac{ar^2}{f(t)}}$$

$$= \frac{r f(t) \frac{\partial f}{\partial t}}{1 - \frac{ar^2}{f(t)}} \{ 1 - 1 \} = 0 = g_{01}$$

Then $g_{02} \overset{!}{=} \tilde{g}_{00} \frac{\partial y^0}{\partial x^0} \underbrace{\frac{\partial y^0}{\partial x^2}}_{=0} + \tilde{g}_{11} \frac{\partial y^1}{\partial x^0} \underbrace{\frac{\partial y^1}{\partial x^2}}_{=0} + \tilde{g}_{22} \underbrace{\frac{\partial y^2}{\partial x^0}}_{=0} \frac{\partial y^2}{\partial x^2} + \tilde{g}_{33} \underbrace{\frac{\partial y^3}{\partial x^0}}_{=0} \frac{\partial y^3}{\partial x^2}$

$= 0 = g_{02}$

Next $g_{03} \overset{!}{=} \tilde{g}_{00} \frac{\partial y^0}{\partial x^0} \underbrace{\frac{\partial y^0}{\partial x^3}}_{=0} + \tilde{g}_{11} \frac{\partial y^1}{\partial x^0} \underbrace{\frac{\partial y^1}{\partial x^3}}_{=0} + \tilde{g}_{22} \underbrace{\frac{\partial y^2}{\partial x^0}}_{=0} \frac{\partial y^2}{\partial x^3} + \tilde{g}_{33} \underbrace{\frac{\partial y^3}{\partial x^0}}_{=0} \frac{\partial y^3}{\partial x^3}$

$= 0 = g_{03}$

and $g_{12} \overset{!}{=} \tilde{g}_{00} \frac{\partial y^0}{\partial x^1} \underbrace{\frac{\partial y^0}{\partial x^2}}_{=0} + \tilde{g}_{11} \frac{\partial y^1}{\partial x^1} \underbrace{\frac{\partial y^1}{\partial x^2}}_{=0} + \tilde{g}_{22} \underbrace{\frac{\partial y^2}{\partial x^1}}_{=0} \frac{\partial y^2}{\partial x^2} + \tilde{g}_{33} \underbrace{\frac{\partial y^3}{\partial x^1}}_{=0} \frac{\partial y^3}{\partial x^2}$

$= 0 = g_{12}$

$$g_{13} \overset{!}{=} \tilde{g}_{00} \frac{\partial y^0}{\partial x^1} \underbrace{\frac{\partial y^0}{\partial x^3}}_{=0} + \tilde{g}_{11} \frac{\partial y^1}{\partial x^1} \underbrace{\frac{\partial y^1}{\partial x^3}}_{=0} + \tilde{g}_{22} \underbrace{\frac{\partial y^2}{\partial x^1}}_{=0} \frac{\partial y^2}{\partial x^3} + \tilde{g}_{33} \underbrace{\frac{\partial y^3}{\partial x^1}}_{=0} \frac{\partial y^3}{\partial x^3}$$

$$= 0 = g_{13}$$

$$g_{23} \overset{!}{=} \tilde{g}_{00} \frac{\partial y^0}{\partial x^2} \underbrace{\frac{\partial y^0}{\partial x^3}}_{=0} + \tilde{g}_{11} \frac{\partial y^1}{\partial x^2} \underbrace{\frac{\partial y^1}{\partial x^3}}_{=0} + \tilde{g}_{22} \frac{\partial y^2}{\partial x^2} \underbrace{\frac{\partial y^2}{\partial x^3}}_{=0} + \tilde{g}_{33} \underbrace{\frac{\partial y^3}{\partial x^2}}_{=0} \frac{\partial y^3}{\partial x^3}$$

$$= 0 = g_{23}$$

c) We have

$$\bar{t}(t,r) = \left(\frac{1-ar_0^2}{a}\right)^{1/2} \int_{S(t,r)}^{1} d\rho \, \frac{\left(\frac{\rho}{1-\rho}\right)^{1/2}}{1-\frac{ar_0^2}{\rho}}$$

2.5

where $S(t,r) = 1 - \left(\frac{1-ar^2}{1-ar_0^2}\right)^{1/2}(1-f(t))$

s.t. $S(t,r_0) = f(t)$

First I looked at $\frac{\partial \bar{t}}{\partial t}\big|_{t_s,r_0}$ and $\frac{\partial \bar{t}}{\partial r}\big|_{t_s,r_0}$ which were infinity. Could one also conclude that for $f(x)$ with $f'(x)|_{x=\rho}=\infty$ also $f(x)|_{x=\rho}=\infty$?

no. eg:- $x^{1/2}|_{x=0}=0$ but $(x^{1/2})'|_{x=0} \to \infty$.

We now take t_s with $r_0 f(t_s) = r_g$ and from a) we take $r_g = ar_0^3$

$$\Rightarrow f(t_s) = ar_0^2$$

Why $r_g = ar_0^3$? Only showed in (a) that for $r_g = ar_0^3$ A and B take the special form?

Yes, I think we want the two metrics to be equal for $r=r_0$ and thus from a) $r_g = ar_0^3$.

Then $\bar{t}(t_s,r_0) = \left(\frac{1-ar_0^2}{a}\right)^{1/2} \int_{f(t_s)}^{1} d\rho \, \frac{\left(\frac{\rho}{1-\rho}\right)^{1/2}}{1-\frac{ar_0^2}{\rho}}$

$$= \left(\frac{1-ar_0^2}{a}\right)^{1/2} \int_{ar_0^2}^{1} d\rho \, \underbrace{\frac{\left(\frac{\rho}{1-\rho}\right)^{1/2}}{1-\frac{ar_0^2}{\rho}}}_{F(\rho)}$$

which obviously has a pole at the lower integration bound s.t. the integrand becomes infinity $F(\rho) \xrightarrow{\rho = ar_0^2} \infty$ ~~and thus~~ $\bar{t}(t_s,r_0) = \infty$

Counter Example
$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \left[\frac{x^{1/2}}{1/2}\right]_0^1 = 2$$
but integrand $\to \infty$ at $x=0$.

If instead, the lower integration bound would be above $\rho = ar_0^2$, we would not "hit" the pole, i.e. we need "$\rho > ar_0^2$" or "$f(t) > ar_0^2$". In the lecture, we found from the differential equation for $f(t)$ that $f(t)$ is monotonically decreasing with t s.t. $t_s > t$ gives $f(t) > ar_0^2 = f(t_s)$. Also, $f(t) \leq 1$ (the upper bound). The initial values were $f(0)=1$, $\dot{f}(0)=0$ s.t. $t \geq 0$ guarantees $f(t) \leq 1$

Why is $f(t)$ (always) a decreasing function of t?

Look at differential equation. Think of time direction $t \geq 0$.

$$\Rightarrow 0 \leq t < t_s$$

$f'(0)=0$ but $f''(0)<0$ so decreases

$$\Rightarrow f(t_s) < f(t) \leq f(0) \Leftrightarrow r_g < r_0 f(t) \leq r_0$$

d) Having $0 \le t < t_s$ and $\frac{r_g}{f(t)} < r \le r_0$, we use now

$$\bar{r}(t,r) = r f(t)$$

$$\bar{t}(t,r) = \left(\frac{1-ar_0^2}{a}\right)^{1/2} \int_{S(t,r)}^{1} d\rho \frac{\left(\frac{\rho}{1-\rho}\right)^{1/2}}{1-\frac{ar_0^2}{\rho}}$$

Why can we fix $\bar{r} = r_0$ here to find $\bar{t} < \infty$? Only valid for r_0? (only possibility)

2

From c), we already know that $\bar{t}(t_s, r_0) = \infty$ s.t. $\bar{t} < \infty$ for $t < t_s$.

It's also clear that $S(t=0,r) = 1$ ($f(0)=1$), s.t.

$$\bar{t}(t=0,r) = 0$$

$$\rightsquigarrow \quad 0 \le \bar{t} < \infty$$

With the help of the other equation, we then find

Why do we need $f(t(\bar{t}))$ with $t(\bar{t})$ implicitly defined as on the sheet? ($\bar{t}(t)$ so, inverse $\Rightarrow t(\bar{t})$)

$$\bar{r}(t,r) = r f(t) \le r_0 f(t) = r_0 f(t(\bar{t}))$$

and $\bar{r}(t,r) = r f(t) > r_g$

$$\rightsquigarrow \quad r_g < \bar{r}(t,r) \le r_0 f(t(\bar{t}))$$

It's $\bar{r} > r_g$, not $\bar{r} \ge r_g$? (sloppy sheet)

2

e) We thus constructed a map $\hat{\varphi}: D \to \bar{D}$, for which

- $g = \hat{\varphi}^* \bar{g}$ as we found in (b)
- $\tilde{g}_{\alpha\beta} = \bar{g}_{\alpha\beta}$ for $\bar{r} = r_0 f(t) = \bar{r}(t,r)$ and $\bar{t} = t(t, r_0)$, i.e. for $r = r_0$ as shown in (a) for $A(t,r_0)$, $B(t,r_0)$ which are then used to define $\tilde{g}$ and matching it to $\bar{g}$; for $\tilde{g}_{22}, \tilde{g}_{33}$ and $\bar{g}_{22}, \bar{g}_{33}$ this follows instantly with $\bar{r}^2(t,r_0) = r_0^2 f^2(t)$
- the domain of $\hat{\varphi}$ is $D \to \bar{D}$ which we proved in (c) and (d)

$\rightsquigarrow$ We successfully constructed a mapping from the cloud's comoving coordinates x^μ onto the Schwarzschild coordinates, for which also the metric $\tilde{g}$ – constructed on these sheets and which is equal to the Schwarzschild metric $\bar{g}$ at the border of the cloud r_0 – is mapped onto the interior dust cloud metric g (given in terms of the clouds comoving coordinates x^μ).