Disclaimer

General Relativity 9. Home exercise — Manni Zauba

18.06.2018 H2) Comoving coordinates $x^0 = t$, $x^1 = r$, $x^2 = \vartheta$, $x^3 = \phi$ and metric

2/2

$$g_{00} = 1, \quad g_{11} = \frac{-f(t)^2}{1-ar^2}$$

But in comoving coord. only $g_{ii} \neq 0$, but necessarily diagonal?

$$g_{22} = -f(t)^2 r^2, \quad g_{33} = -f(t)^2 r^2 \sin^2\vartheta$$

for the non-vanishing components

$g_{0i} = 0$ ~~$\neq 0$~~

g_{ij} could also be non-diag. → more complicated

a) consider coord. trafo φ: $y^0 = t = x^0$, $y^1 = \chi$

$y^2 = \theta = x^2$

$y^3 = \phi = x^3$

Why the $r(\chi)$? - Just a "re"-parametrization?

Y_g

where $\quad r(\chi) = \frac{1}{\sqrt{a}} \sin\chi \Leftrightarrow \chi = \sin^{-1}(\sqrt{a}\, r)$

We compute $\tilde{g}_{\alpha\beta} = (\varphi^* g)_{\alpha\beta}$:

$$(\varphi^* g)_{\alpha\beta}(x) = g_{\mu\nu}(\varphi(x)) \frac{\partial \varphi^\mu}{\partial x^\alpha} \frac{\partial \varphi^\nu}{\partial x^\beta} \text{ for } x \mapsto \varphi(x)$$

$$\rightsquigarrow (\varphi^* g)_{\alpha\beta}(y) = g_{\mu\nu}(x) \frac{\partial x^\mu}{\partial y^\alpha} \frac{\partial x^\nu}{\partial y^\beta}$$

as here we have $g_{\mu\nu}(x) = g_{\mu\nu}(\varphi(y))$ given

and thus $\varphi: y \mapsto \varphi(y) = x$

We instantly notice that $\frac{\partial x^\mu}{\partial y^\nu} = \delta^\mu{}_\nu$ for $\mu (= \nu) \neq 1$

and $\frac{\partial x^1}{\partial y^1} = \frac{\partial r}{\partial \chi} = \frac{1}{\sqrt{a}} \cos\chi$

while all other derivatives vanish

What if $f(t) = f(x^0)$ for $x^0 = t$ but new coord. $y^0 \neq t$? Then express t through $y^0, y^1, \ldots$? Not always possible, different f?

$$\rightsquigarrow \tilde{g}_{\alpha\beta}(y) = g_{00}(x) \frac{\partial x^0}{\partial y^\alpha} \frac{\partial x^0}{\partial y^\beta} + g_{11}(x) \frac{\partial x^1}{\partial y^\alpha} \frac{\partial x^1}{\partial y^\beta} + g_{22}(x) \frac{\partial x^2}{\partial y^\alpha} \frac{\partial x^2}{\partial y^\beta} + g_{33}(x) \frac{\partial x^3}{\partial y^\alpha} \frac{\partial x^3}{\partial y^\beta}$$

$$= \delta^0_\alpha \delta^0_\beta - \delta^1_\alpha \delta^1_\beta \frac{\cos^2\chi}{a} \frac{f(t)^2}{1 - ar(\chi)^2} - \delta^2_\alpha \delta^2_\beta f(t)^2 r^2(\chi)$$

$$- \delta^3_\alpha \delta^3_\beta f(t)^2 r(\chi)^2 \sin^2\vartheta$$

$$= \delta^0_\alpha \delta^0_\beta - \delta^1_\alpha \delta^1_\beta \frac{f(t)^2}{a} - \delta^2_\alpha \delta^2_\beta \frac{f(t)^2 \sin^2\chi}{a} - \delta^3_\alpha \delta^3_\beta \frac{f(t)^2 \sin^2\chi \sin^2\vartheta}{a}$$

with $\tilde{g}_{00} = 1$, $\tilde{g}_{11} = -f(t)^2/a$, $\tilde{g}_{22} = -\frac{f(t)^2}{a} \sin^2\psi$

$\tilde{g}_{33} = -\frac{f(t)^2}{a} \sin^2\psi \sin^2\theta$

(1)

b) We define $n \in \mathbb{R}^4$ by $n^1 = \cos\psi$, $n^2 = \sin\psi \sin\theta \cos\phi$

$n^3 = \sin\psi \sin\theta \sin\phi$, $n^4 = \sin\psi \cos\theta$, no $t = y^0$ dependence!

(?)

We first calculate: $\frac{\partial n^1}{\partial y^1} = -\sin\psi$, $\frac{\partial n^1}{\partial y^k} = 0$ for $k = 2,3$

$\frac{\partial n^2}{\partial y^1} = \cos\psi \sin\theta \cos\phi$, $\frac{\partial n^2}{\partial y^2} = \sin\psi \cos\theta \cos\phi$, $\frac{\partial n^2}{\partial y^3} = -\sin\psi \sin\theta \sin\phi$

$\frac{\partial n^3}{\partial y^1} = \cos\psi \sin\theta \sin\phi$, $\frac{\partial n^3}{\partial y^2} = \sin\psi \cos\theta \sin\phi$, $\frac{\partial n^3}{\partial y^3} = \sin\psi \sin\theta \cos\phi$

$\frac{\partial n^4}{\partial y^1} = \cos\psi \cos\theta$, $\frac{\partial n^4}{\partial y^2} = -\sin\psi \sin\theta$, $\frac{\partial n^4}{\partial y^3} = 0$

We define $\bar{g}_{00} = 1$, $\bar{g}_{ik} = -\frac{f(t)^2}{a} \sum_{\alpha=1}^{4} \frac{\partial n^\alpha}{\partial y^i} \frac{\partial n^\alpha}{\partial y^k}$, $\bar{g}_{0i} = 0$

$\bar{g}_{ik}$ is obviously symmetric under $i \leftrightarrow k$ s.t. we will only calculate $(1,2), (1,3), (2,3)$ and $(1,1), (2,2), (3,3)$

$$\bar{g}_{11} = -\frac{f(t)^2}{a} \Big\{ \sin^2\psi + \underbrace{\underbrace{\cos^2\psi \sin^2\theta \cos^2\phi + \cos^2\psi \sin^2\theta \sin^2\phi}_{\cos^2\psi \sin^2\theta} + \cos^2\psi \cos^2\theta}_{\cos^2\psi} \Big\}$$

$$= -\frac{f(t)^2}{a}$$

$$\bar{g}_{22} = -\frac{f(t)^2}{a} \Big\{ \sin^2\psi \cos^2\theta \cos^2\phi + \sin^2\psi \cos^2\theta \sin^2\phi + \sin^2\psi \sin^2\theta \Big\} = -\frac{f(t)^2}{a} \sin^2\psi$$

$$\bar{g}_{33} = -\frac{f(t)^2}{a} \Big\{ \sin^2\psi \sin^2\theta \sin^2\phi + \sin^2\psi \sin^2\theta \cos^2\phi \Big\} = -\frac{f(t)^2}{a} \sin^2\psi \sin^2\theta$$

$$\bar{g}_{12} = -\frac{f(t)^2}{a} \Big\{ \sin\psi \cos\psi \sin\theta \cos\theta \cos^2\phi + \sin\psi \cos\psi \sin\theta \cos\theta \sin^2\phi - \sin\psi \cos\psi \sin\theta \cos\theta \Big\} = 0$$

$$\bar{g}_{13} = -\frac{f(t)^2}{a} \Big\{ -\sin\psi \cos\psi \sin^2\theta \sin\phi \cos\phi + \sin\psi \cos\psi \sin^2\theta \sin\phi \cos\phi \Big\} = 0$$

$$\bar{g}_{23} = -\frac{f(t)^2}{a} \Big\{ -\sin^2\psi \sin\theta \cos\theta \sin\phi \cos\phi + \sin^2\psi \sin\theta \cos\theta \sin\phi \cos\phi \Big\} = 0$$

~~Interpretation?~~ Intuit.

Thus, we have $\bar{g} = \tilde{g}$. This means that in the coordinate system spanned by n^i, $i = 1,2,3,4$, the metric has the simple form

$$\begin{pmatrix} 1 & & & 0 \\ & -\frac{f(t)^2}{a} & & \\ & & -\frac{f(t)^2}{a} & \\ 0 & & & -\frac{f(t)^2}{a} \end{pmatrix}$$