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General Relativity J. Home exercise

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18.06.2018 4P) Comoving coordinates $x^0 = t, x^1 = r, x^2 = \vartheta, x^3 = \phi$ and metric

$$g_{00} = 1, \quad g_{11} = \frac{-f(t)^2}{1 - ar^2}$$

$$g_{22} = -f(t)^2 r^2, \quad g_{33} = -f(t)^2 r^2 \sin^2 \vartheta$$

2/2

But in comoving coord. only g_{00} is diagonal?

for the non-vanishing components

$g_{0i} = 0$
~~also~~

we could also be non-diag. \rightarrow more complicated

a) consider coord. transform $\varphi: y^0 = t = x^0, y^1 = \vartheta, y^2 = \theta = x^2, y^3 = \phi = x^3$

where $r(\vartheta) = \frac{1}{\sqrt{a}} \sin \vartheta \Leftrightarrow \vartheta = \sin^{-1}(\sqrt{a}r)$

Why the $r(\vartheta)$?
 - just a re-parametrization?

We compute $\tilde{g}_{\alpha\beta} = (\varphi^* g)_{\alpha\beta}$:

$$(\varphi^* g)_{\alpha\beta}(y) = g_{\mu\nu}(\varphi(y)) \frac{\partial x^\mu}{\partial y^\alpha} \frac{\partial x^\nu}{\partial y^\beta} \text{ for } x \mapsto \varphi(y)$$

$$\rightarrow (\varphi^* g)_{\alpha\beta}(y) = g_{\mu\nu}(x) \frac{\partial x^\mu}{\partial y^\alpha} \frac{\partial x^\nu}{\partial y^\beta}$$

as here we have $g_{\mu\nu}(x) = g_{\mu\nu}(\varphi(y))$ given and thus $\varphi: y \mapsto \varphi(y) = x$

We instantly notice that $\frac{\partial x^\mu}{\partial y^\nu} = \delta^\mu_\nu$ for $\mu (= \nu) \neq 1$

$$\text{and } \frac{\partial x^1}{\partial y^1} = \frac{\partial r}{\partial \vartheta} = \frac{1}{\sqrt{a}} \cos \vartheta$$

while all other derivatives vanish

What if $f(t) = f(x)$ for $x^0 = t$ but new coord. $y^0 \neq t$? then express t through y^0, y^1, \dots ?

Not always possible, different f ?

$$\rightarrow \tilde{g}_{\alpha\beta}(y) = g_{00}(x) \frac{\partial x^0}{\partial y^\alpha} \frac{\partial x^0}{\partial y^\beta} + g_{11}(x) \frac{\partial x^1}{\partial y^\alpha} \frac{\partial x^1}{\partial y^\beta} + g_{22}(x) \frac{\partial x^2}{\partial y^\alpha} \frac{\partial x^2}{\partial y^\beta} + g_{33}(x) \frac{\partial x^3}{\partial y^\alpha} \frac{\partial x^3}{\partial y^\beta}$$

$$= \delta^0_\alpha \delta^0_\beta - \delta^1_\alpha \delta^1_\beta \frac{\cos^2 \vartheta}{a} \frac{f(t)^2}{1 - ar^2} - \delta^2_\alpha \delta^2_\beta f(t)^2 r^2 \sin^2 \vartheta$$

$$= \delta^0_\alpha \delta^0_\beta - \delta^1_\alpha \delta^1_\beta \frac{f(t)^2}{a} - \delta^2_\alpha \delta^2_\beta \frac{f(t)^2 \sin^2 \vartheta}{a} - \delta^3_\alpha \delta^3_\beta \frac{f(t)^2 \sin^2 \vartheta \sin^2 \vartheta}{a}$$

$$\tilde{g}_{00} = 1, \quad \tilde{g}_{11} = -\frac{f(\varphi)^2}{a}, \quad \tilde{g}_{22} = -\frac{f(\varphi)^2}{a} \sin^2 \varphi$$

$$\tilde{g}_{33} = -\frac{f(\varphi)^2}{a} \sin^2 \varphi \sin^2 \theta$$

(1)

b) We define $n \in \mathbb{R}^4$ by $n^1 = \cos \varphi$, $n^2 = \sin \varphi \sin \theta \cos \phi$
 $n^3 = \sin \varphi \sin \theta \sin \phi$, $n^4 = \sin \varphi \cos \theta$, no $t = y^0$ dependence!

(1)

We first calculate: $\frac{\partial n^1}{\partial y^2} = -\sin \varphi$, $\frac{\partial n^1}{\partial y^k} = 0$ for $k=2,3$

$$\frac{\partial n^2}{\partial y^1} = \cos \varphi \sin \theta \cos \phi, \quad \frac{\partial n^2}{\partial y^2} = \sin \varphi \cos \theta \cos \phi, \quad \frac{\partial n^2}{\partial y^3} = -\sin \varphi \sin \theta \sin \phi$$

$$\frac{\partial n^3}{\partial y^1} = \cos \varphi \sin \theta \sin \phi, \quad \frac{\partial n^3}{\partial y^2} = \sin \varphi \cos \theta \sin \phi, \quad \frac{\partial n^3}{\partial y^3} = \sin \varphi \sin \theta \cos \phi$$

$$\frac{\partial n^4}{\partial y^0} = \cos \varphi \cos \theta, \quad \frac{\partial n^4}{\partial y^2} = -\sin \varphi \sin \theta, \quad \frac{\partial n^4}{\partial y^3} = 0$$

We define $\bar{g}_{00} = 1$, $\bar{g}_{ik} = -\frac{f(\varphi)^2}{a} \sum_{\alpha=1}^4 \frac{\partial n^\alpha}{\partial y^i} \frac{\partial n^\alpha}{\partial y^k}$, $\bar{g}_{0i} = 0$

\bar{g}_{ik} is obviously symmetric under $i \leftrightarrow k$ s.t. we will only

calculate $(1,2)$, $(1,3)$, $(2,3)$ and $(1,1)$, $(2,2)$, $(3,3)$

$$\bar{g}_{11} = -\frac{f(\varphi)^2}{a} \left\{ \sin^2 \varphi + \underbrace{\cos^2 \varphi \sin^2 \theta \cos^2 \phi + \cos^2 \varphi \sin^2 \theta \sin^2 \phi + \cos^2 \varphi \cos^2 \theta}_{\cos^2 \varphi \sin^2 \theta} \right\}$$

$$= -\frac{f(\varphi)^2}{a}$$

$$\bar{g}_{22} = -\frac{f(\varphi)^2}{a} \left\{ \sin^2 \varphi \cos^2 \theta \cos^2 \phi + \sin^2 \varphi \cos^2 \theta \sin^2 \phi + \sin^2 \varphi \sin^2 \theta \right\} = -\frac{f(\varphi)^2}{a} \sin^2 \varphi$$

$$\bar{g}_{33} = -\frac{f(\varphi)^2}{a} \left\{ \sin^2 \varphi \sin^2 \theta \sin^2 \phi + \sin^2 \varphi \sin^2 \theta \cos^2 \phi \right\} = -\frac{f(\varphi)^2}{a} \sin^2 \varphi \sin^2 \theta$$

$$\bar{g}_{12} = -\frac{f(\varphi)^2}{a} \left\{ \sin \varphi \cos \varphi \sin \theta \cos \theta \cos^2 \phi + \sin \varphi \cos \varphi \sin \theta \cos \theta \sin^2 \phi \right. \\ \left. - \sin \varphi \cos \varphi \sin \theta \cos \theta \right\} = 0$$

$$\bar{g}_{13} = -\frac{f(\varphi)^2}{a} \left\{ -\sin \varphi \cos \varphi \sin^2 \theta \sin \phi \cos \phi + \sin \varphi \cos \varphi \sin^2 \theta \sin \phi \cos \phi \right\} = 0$$

$$\bar{g}_{23} = -\frac{f(\varphi)^2}{a} \left\{ -\sin^2 \varphi \sin \theta \cos \theta \sin \phi \cos \phi + \sin^2 \varphi \sin \theta \cos \theta \sin \phi \cos \phi \right\} = 0$$

Interpretation?

Induct.

Thus, we have $\bar{g} = \tilde{g}$. This means that in the coordinate system

spanned by $n^i, i=1,2,3,4$, the metric has the simple form

$$\begin{pmatrix} 1 & & & \\ & -\frac{f(\varphi)^2}{a} & & \\ & & -\frac{f(\varphi)^2}{a} & \\ & & & -\frac{f(\varphi)^2}{a} \end{pmatrix}$$