Disclaimer

The solution at hand was written in the course of the respective class at the University of Bonn. If not stated differently on top of the first page or the following website, the solution was prepared and handed in solely by me, Marvin Zanke. Anything in a different color than the ball pen blue is usually a correction that I or a tutor made. For more information and all my material, check:
https://www.physics-and-stuff.com/

I raise no claim to correctness and completeness of the given solutions! This equally applies to the corrections mentioned above.

13.10.2017 Group theory 1st Exercise Homework

H 1.1 $G = \{2, 4, 6, 8\}$ and $(G, \circ)$ where

$g_1 \circ g_2 := g_1 g_2 \bmod 10$

why use G and not just $\mathbb{Z}_4$?

(1)

	2	4	6	8
2	4	8	2	6
4	8	6	4	2
6	2	4	6	8
8	6	2	8	4

+

obviously t_{ij} is symmetric, which we expected, as the Group inherits the abelian structure from the multiplication of integer numbers

(2) neutral element $e = 6$, which can easily be seen from the multiplication table +

(3) We then look for $g_1 g_2 = e$ (6) in the table:

g	g^{-1}
2	8
4	4
6	6
8	2

+

(4)
+ $2^4 = 16 \bmod 10 = 6 \rightsquigarrow \text{ord}(2) = 4$
+ $4^2 = 16 \bmod 10 = 6 \rightsquigarrow \text{ord}(4) = 2$
− $6^2 = 36 \bmod 10 = 6 \rightsquigarrow \cancel{\text{ord}(6) = 2}$ $\text{ord}(6) = 1$. $6 = e$
+ $8^4 = 4096 \bmod 10 = 6 \rightsquigarrow \text{ord}(8) = 4$

and $|G| = 4$

$\rightsquigarrow 0 = |G| \bmod \text{ord}(g)$ (Lagrange's theorem)

Better way?

(5) G is isomorphic to C_4:

	e	a	a^2	a^3
e	e	a	a^2	a^3
a	a	a^2	a^3	e
a^2	a^2	a^3	e	a
a^3	a^3	e	a	a^2

+

$\phi : C_4 \to G$

$\phi(e) = 6$ neutral element
$\phi(a^2) = 4$ other self inverse element
$\left.\begin{array}{l}\phi(a) = 2 \\ \phi(a^3) = 8\end{array}\right\}$ interchangeable

$\phi(a) = 8$ and $\phi(a^3) = 2$ really an interchangeable set?

10 p

H 1.2

(1) The groups with 1 element $G_1 = \{e\}$ and 2 elements $G_2 = \{e, a\}$ are abelian by triviality.

Let's consider a group G of 3 or more elements with a neutral element e. Let's assume $e \neq a, b \in G$ with $a \neq b$, $ab \neq ba$. We know, that $ab \in G$ and $ba \in G$ by definition.

But:
- $ab = e$ isn't possible, as this would imply $a^{-1} = b$ and make a, b commute in contradiction to assumption.
- $ab = a$ and $ab = b$ isn't possible as well, as this would make $b = e$ or $a = e$

$\Longrightarrow ab = c \in G$

Furthermore, we assumed $ba \neq ab$ and using the same arguments as above, we also find that $ba = d \in G$ is a new element in the group. We thus need 5 elements in a group for the group to be non-abelian, making all groups with at most 4 elements abelian.

(2) First, we will show that there are only 2 distinct groups with 4 elements.

	e	a	b	c
e	e	a	b	c
a	a			
b	b			
c	c			

- We now have 3 possibilities for a^2: e, b, c. By relabeling, we can consider $b \leftrightarrow c$ as equivalent solutions, leaving us with 2 possibilities:

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c		
c	c	b		

(*1)

	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

(*2)

?

using only the rearrangement theorem A-1.3

For the first table (*1), we furthermore have 2 possibilities now: $b^2 = e$ and $b^2 = a$:

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

1^{st}

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	a	e
c	c	b	e	a

2^{nd}

using the rearrangement theorem again.

Taking a closer look at (*2), we rearrange the group to $\{e, b, a, c\}$, making the table look like

(change the rows and columns $a \leftrightarrow b$)

	e	b	a	c
e	e	b	a	c
b	b	e	c	a
a	a	c	b	e
c	c	a	e	b

Renaming $a \leftrightarrow b$ yields the 2^{nd} group structure from above and thus the two tables at the top are the only possible $|G| = 4$ groups

Now take a look at $\mathbb{Z}_4$:

	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

and $\mathbb{Z}_2 \times \mathbb{Z}_2$:

	(0,0)	(1,0)	(0,1)	(1,1)
(0,0)	(0,0)	(1,0)	(0,1)	(1,1)
(1,0)	(1,0)	(0,0)	(1,1)	(0,1)
(0,1)	(0,1)	(1,1)	(0,0)	(1,0)
(1,1)	(1,1)	(0,1)	(1,0)	(0,0)

Which obviously are two different group structures with 4 elements – in one, each element is its own inverse, and in the other, it's one of the other elements; just like in the general case on top of the page. Thus, one can easily construct an isomorphism between those groups, making the 1^{st} group (table) isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ and the 2^{nd} group to $\mathbb{Z}_4$.

+

(3) $\mathbb{Z}_4 \simeq C_4$ Rotation group of a square (directed)

$\mathbb{Z}_2 \times \mathbb{Z}_2 \simeq D_2$ Symmetry group of a rectangle (undirected)

analogy from abstract groups to point sym. and geometry?

10p

+

Total: Maximum

M.M.