Disclaimer

The solution at hand was written in the course of the respective class at the University of Bonn. If not stated differently on top of the first page or the following website, the solution was prepared and handed in solely by me, Marvin Zanke. Anything in a different color than the ball pen blue is usually a correction that I or a tutor made. For more information and all my material, check:

https://www.physics-and-stuff.com/

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Group Theory 1st Exercise Homework

H.1.1 \( G = \{2, 4, 6, 8\} \) and \((G, \ast)\) where

\[
G \ast G = G, \quad g_1 \ast g_2 \mod 10
\]

\[
\begin{array}{cccc}
2 & 4 & 6 & 8 \\
4 & 8 & 2 & 6 \\
6 & 2 & 4 & 8 \\
8 & 6 & 2 & 4 \\
\end{array}
\]

Why use \( G \) and not just \( \mathbb{Z}_4 \)?

\( 1 \) Obviously \( (G, \ast) \) is symmetric, which we expected, as the group inherits the abelian structure from the multiplication of integer numbers.

\( 2 \) Neutral element \( e = 6 \), which can easily be seen from \( 6 \ast 6 = e^6 \) in the table:

\[
\begin{array}{c|c}
G & G \\
\hline
8 & 8 \\
2 & 8 \\
4 & 4 \\
6 & 6 \\
8 & 2 \\
\end{array}
\]

\( 3 \) We then look for \( g_1 \ast g_2 = e^6 \) in the table:

\[
\begin{array}{c}
8 \\
2 \\
4 \\
6 \\
8 \\
\end{array}
\]

\( 4 \) \( 2^4 = 16 \mod 10 = 6 \implies \text{ord}(2) = 4 \)

\( 4^2 = 16 \mod 10 = 6 \implies \text{ord}(4) = 2 \)

\( 6^2 = 36 \mod 10 = 6 \implies \text{ord}(6) = 2 \)

\( 8^4 = 4096 \mod 10 = 6 \implies \text{ord}(8) = 4 \)

\( 0 \equiv 16l \mod \text{ord}(8) \) (Lagrange’s theorem)

\( 5 \) \( G_1 \) is isomorphic to \( C_4 \):

- \( e \)
- \( a \)
- \( a^2 \)
- \( a^3 \)

\[
\begin{array}{c|cccc}
\phi & e & a & a^2 & a^3 \\
\hline
\phi(e) & e & a & a^2 & a^3 \\
\phi(a) & a^2 & a & e & a^3 \\
\phi(a^2) & a & a^2 & e & a^3 \\
\phi(a^3) & a^3 & e & a & a^2 \\
\end{array}
\]

Better way? \( \phi(a) = 8 \) and \( \phi(a^2) = 8 \) really an \( \mathbb{Z}_8 \) homomorphism?
H.2

1. The groups with 1 element $G_1 = \{ e \}$ and 2 elements $G_2 = \{ e, a \}$ are abelian by triviality.

Let's consider a group of 3 or more elements with a neutral element $e$. Let's assume $e, a, b \in G$ with $ab \neq ba$. We know that $ab \in G$ and $ba \in G$ by definition. But if $ab = c$ isn't possible, as this would imply $a^{-1}b = b$ and make $a, b$ commute in contradiction to assumption. If $ab = a$ and $ab = b$ isn't possible as well, as this would make $b = e$ or $a = e$.

$\Rightarrow ab = c \in G$

Furthermore, we assumed $ba = ab$ and using the same arguments as above, we find that $ba = c \in G$ is a new element in the group. To thus need 5 elements in a group for the group to be non-abelian, making all groups with at most 4 elements abelian.

2. First, we will show that there are only 2 distinct groups with 4 elements.

<table>
<thead>
<tr>
<th>e</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>e</td>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>a</td>
<td>b</td>
<td>c</td>
<td>e</td>
</tr>
<tr>
<td>b</td>
<td>c</td>
<td>e</td>
<td>a</td>
</tr>
<tr>
<td>c</td>
<td>e</td>
<td>a</td>
<td>b</td>
</tr>
</tbody>
</table>

We now have 3 possibilities for $a^2$: $e, b, c$. By relabeling, we can consider $b \rightarrow c$ as equivalent solutions, leaving us with 2 possibilities:

<table>
<thead>
<tr>
<th>e</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>e</td>
<td>a</td>
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<td>c</td>
</tr>
<tr>
<td>e</td>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
</tbody>
</table>

Using only the rearrangement theorem A. 1.3
For the first table, we furthermore have 2 possibilities now: \( b^2 = e \) and \( b^2 = a \).

\[
\begin{array}{|c|c|c|c|}
\hline
\mathbf{e} & \mathbf{a} & \mathbf{b} & \mathbf{c} \\
\hline
\mathbf{a} & \mathbf{e} & \mathbf{c} & \mathbf{b} \\
\mathbf{b} & \mathbf{c} & \mathbf{a} & \mathbf{e} \\
\mathbf{c} & \mathbf{b} & \mathbf{a} & \mathbf{e} \\
\hline
\end{array}
\qquad
\begin{array}{|c|c|c|c|}
\hline
\mathbf{e} & \mathbf{a} & \mathbf{b} & \mathbf{c} \\
\hline
\mathbf{e} & \mathbf{e} & \mathbf{a} & \mathbf{b} \\
\mathbf{a} & \mathbf{a} & \mathbf{e} & \mathbf{b} \\
\mathbf{b} & \mathbf{b} & \mathbf{c} & \mathbf{a} \\
\mathbf{c} & \mathbf{c} & \mathbf{b} & \mathbf{e} \\
\hline
\end{array}
\]

Using the rearrangement theorem again.

Taking a closer look at (12), we rearrange the group to \( \{ e, b, a, c \} \), making the table look like:

\[
\begin{array}{|c|c|c|c|}
\hline
\mathbf{e} & \mathbf{b} & \mathbf{a} & \mathbf{c} \\
\hline
\mathbf{e} & \mathbf{e} & \mathbf{b} & \mathbf{c} \\
\mathbf{b} & \mathbf{b} & \mathbf{a} & \mathbf{c} \\
\mathbf{a} & \mathbf{a} & \mathbf{c} & \mathbf{b} \\
\mathbf{c} & \mathbf{c} & \mathbf{b} & \mathbf{a} \\
\hline
\end{array}
\]

Changing the rows and columns \( a \leftrightarrow b \).

Reversing \( a \leftrightarrow b \) yields the 2nd group structure from above and thus the two tables at the top are the only possible \(|G|=4\) groups.

Now take a look at \( \mathbb{Z}_4 \): \[
\begin{pmatrix}
0 & 1 & 2 & 3 \\
0 & 1 & 2 & 3 \\
1 & 2 & 3 & 0 \\
2 & 3 & 0 & 1 \\
3 & 0 & 1 & 2 \\
\end{pmatrix}
\]

And \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) with:

\[
\begin{array}{|c|c|c|c|}
\hline
(0,0) & (0,1) & (1,0) & (1,1) \\
\hline
(0,0) & (0,1) & (1,0) & (1,1) \\
(0,1) & (0,0) & (1,1) & (1,0) \\
(1,0) & (1,1) & (0,0) & (0,1) \\
(1,1) & (1,0) & (0,1) & (0,0) \\
\hline
\end{array}
\]

Which obviously are two different group structures with 4 elements— in one, each element is its own inverse, and in the other, it's one of the other elements; just like in the general case on top of the page. Thus, one can easily construct an isomorphism between these groups, making the 1st group (table) isomorphic to \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) and the 2nd group to \( \mathbb{Z}_4 \).

(3) \[
\mathbb{Z}_4 \cong \mathbb{Z}_4 \quad \text{Rohrholz: group of a square (directed)}
\]

\[
\mathbb{Z}_2 \times \mathbb{Z}_2 \cong D_3 \quad \text{Symmetry group of a rectangle (undirected)}
\]

Total: Maximum

M.M.