Disclaimer

The solution at hand was written in the course of the respective class at the University of Bonn. If not stated differently on top of the first page or the following website, the solution was prepared and handed in solely by me, Marvin Zanke. Anything in a different color than the ball pen blue is usually a correction that I or a tutor made. For more information and all my material, check:
https://www.physics-and-stuff.com/

I raise no claim to correctness and completeness of the given solutions! This equally applies to the corrections mentioned above.

15.01.2018 **Group theory Exercise 10** Marvin Zanke

H10.1

(1) Spin-3/2 repr. of SU(2)

$$[J_a^{3/2}]_{kl} = \langle 3/2, 5/2-k | J_a | 3/2, 5/2-l \rangle$$

$$[J_+^{3/2}]_{kl} = \langle 3/2, 5/2-k | J_+ | 3/2, 5/2-l \rangle$$

$$J_-^{3/2} = (J_+^{3/2})^\dagger$$

$$[J_3^{3/2}]_{kl} = \langle 3/2, 5/2-k | J_3^{3/2} | 3/2, 5/2-l \rangle$$

$$\langle j m' | J_+ | j m \rangle = \frac{1}{\sqrt{2}} \sqrt{(j+m+1)(j-m)}\, \delta_{m', m+1}$$

$$\rightsquigarrow \quad J_+^{3/2} = \begin{pmatrix} 0 & \sqrt{3/2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & \sqrt{3/2} \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (k, l \text{ go from } 1 \text{ to } 4)$$

$(J_+)^\dagger = J_-$ always?

$$\rightsquigarrow \quad J_-^{3/2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3/2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{3/2} & 0 \end{pmatrix}$$

$$J_1^{3/2} = \frac{1}{\sqrt{2}} (J_+^{3/2} + J_-^{3/2}) = \begin{pmatrix} 0 & \sqrt{3}/2 & 0 & 0 \\ \sqrt{3}/2 & 0 & 1 & 0 \\ 0 & 1 & 0 & \sqrt{3}/2 \\ 0 & 0 & \sqrt{3}/2 & 0 \end{pmatrix}$$

$$J_2^{3/2} = \frac{1}{\sqrt{2}i} (J_+^{3/2} - J_-^{3/2}) = -i \begin{pmatrix} 0 & \sqrt{3}/2 & 0 & 0 \\ -\sqrt{3}/2 & 0 & 1 & 0 \\ 0 & -1 & 0 & \sqrt{3}/2 \\ 0 & 0 & -\sqrt{3}/2 & 0 \end{pmatrix}$$

$$J_3^{3/2} = \begin{pmatrix} 3/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & -3/2 \end{pmatrix}$$

5P

b) $[J_a, O_x] = O_y [\sigma_a]_{yx}/2$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

spin-1/2 repr., $x = 1,2$

$\rightsquigarrow [J_3, O_1] = O_y [\sigma_3]_{y1}/2 = \frac{1}{2} O_1$

$[J_3, O_2] = -\frac{1}{2} O_2$

$\Rightarrow O_1$ corresponds to 1/2 and O_2 to $-1/2$

Have $\langle 3/2, -1/2, \alpha | O_1^{1/2} | 1, -1, \beta \rangle = A$

Why $O_1^{1/2} = O_2^{1/2}$ for the reduced matrix element?

$\overset{\text{Wigner Eckart}}{=} \underbrace{\langle 3/2, -1/2 | 1/2, +1/2, 1, -1 \rangle}_{= \sqrt{1/3}\text{, see CG-table}} \underbrace{\langle 3/2\, \alpha | O_1^{1/2} | 1, \beta \rangle}_{= C}$

$\rightsquigarrow C = \sqrt{3} A$

Consider $\langle 3/2, -3/2, \alpha | O_2^{1/2} | 1, -1, \beta \rangle$

$\overset{\text{Wigner Eckart}}{=} \underbrace{\langle 3/2, -3/2 | 1/2, -1/2, 1, -1 \rangle}_{= 1} \underbrace{\langle 3/2, \alpha | O_2^{1/2} | 1, \beta \rangle}_{= \langle 3/2, \alpha | O_1^{1/2} | 1, \beta \rangle = C}$

as independent of j, s, α, β

$= \sqrt{3} A$

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H10.2

$$[F(U)]_{ij} = \frac{1}{2} Tr[\sigma_i U \sigma_j U^{-1}]$$

Claim: This map is equivalent to the one from H9.2:

$$U\sigma_i U^+ = R_{ji}\sigma_j$$

$\rightarrow$ $U\sigma_i U^+ = R_{ji}\sigma_j \quad | \times \sigma_k$

$U\sigma_i U^+ \sigma_k = R_{ji}\sigma_j\sigma_k \quad | Tr(\_\_)$

$$Tr(U\sigma_i U^+\sigma_k) = R_{ji} Tr(\delta_{jk}\mathbb{1} + i\epsilon_{jkl}\sigma_l)$$

$$1/2\, Tr(U\sigma_i U^+\sigma_k) = R_{ki}$$

$\rightarrow$ This mapping was shown to be surjective.

It was also shown that $R(U_1) = R(U_2)$

$\rightarrow U_1 = \pm U_2$

+

$\rightarrow [F(\mathbb{1})]_{ij} = \frac{1}{2} Tr[\sigma_i\sigma_j] = \delta_{ij}$

$\rightarrow \mathbb{1} \in ker(F)$ and $\forall x: [F(x)]_{ij} = \delta_{ij} \quad x = \pm\mathbb{1}$

$\rightarrow ker F = \{\mathbb{1}, -\mathbb{1}\}$ which is obviously finite and a subgroup of the Lie-group $SU(2)$ and thus a discrete subgroup

A10.3

$([\cdot,\cdot])$
$= [f(\cdot), f(\cdot)]$
still?

$\xRightarrow{}$ $f(l) = \frac{1}{i}\frac{d}{dt} F(\exp(itl))|_{t=0}$ is an isomorphism

+5P

We check what the images of the basis elements give

$(a_i = \frac{1}{2}\sigma_i)$, $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$[f(\tfrac{1}{2}\sigma_k)]_{ij} = \frac{1}{i}\frac{d}{dt}[F(\exp(\tfrac{it}{2}\sigma_k))]_{ij} \quad \text{(leaving out } t\to 0 \text{ for now)}$$

$$= \frac{1}{2i}\frac{d}{dt} Tr\{\sigma_i e^{it/2\sigma_k}\sigma_j e^{-it/2\sigma_k}\}$$

$$= \frac{1}{2i} Tr\{\sigma_i \tfrac{i}{2} e^{it/2\sigma_k}\sigma_k\sigma_j e^{-it/2\sigma_k} - \tfrac{i}{2}\sigma_i e^{it/2\sigma_k}\sigma_j\sigma_k e^{-it/2\sigma_k}\}$$

$$= \frac{1}{4} Tr\{\sigma_i e^{it/2\sigma_k}[\sigma_k,\sigma_j] e^{-it/2\sigma_k}\}$$

$$= \frac{1}{4} \mathrm{Tr}\left\{ \sigma_i e^{it/2 \sigma_k} (2i\epsilon_{kjz}\sigma_z) e^{-it/2\sigma_k} \right\}$$

$$= \frac{i}{2} \mathrm{Tr}\left\{ \epsilon_{kjz} \sigma_i e^{it/2\sigma_k} \sigma_z e^{-it/2\sigma_k} \right\} \overset{?}{=} \frac{i}{2} \epsilon_{kjz} \left[F(e^{it/2\sigma_k}) \right]_{iz}$$

Explain, please, I do not get it.

$$\xrightarrow{t=0} \frac{i}{2} \epsilon_{kjz} [F(\mathbb{1})]_{iz} = \frac{i}{2} \epsilon_{kjz} \mathbb{1}_{iz} = -\frac{i}{2} \epsilon_{kij}$$

$$\rightsquigarrow \left[f(\tfrac{1}{2}\sigma_1) \right]_{ij} = -\tfrac{i}{\cancel{2}} \epsilon_{1ij} \longrightarrow f(\tfrac{1}{2}\sigma_1) = -\tfrac{i}{\cancel{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \hat{=} X_1$$

$$\left[f(\tfrac{1}{2}\sigma_2) \right]_{ij} = -\tfrac{i}{\cancel{2}} \epsilon_{2ij} \Rightarrow f(\tfrac{1}{2}\sigma_2) = -\tfrac{i}{\cancel{2}} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \hat{=} X_2$$

$$\left[f(\tfrac{1}{2}\sigma_3) \right]_{ij} = -\tfrac{i}{\cancel{2}} \epsilon_{3ij} \rightarrow f(\tfrac{1}{2}\sigma_3) = -\tfrac{i}{\cancel{2}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \hat{=} X_3$$

SO(3) are the matrices fulfilling $O^T O = \mathbb{1}$ and $\det O = 1$

$\rightsquigarrow$ For the generators of the Lie algebra, we have

- $\det(e^{i\alpha_i X_i}) = e^{i\alpha_i \mathrm{tr}(X_i)} \overset{!}{=} 1 \rightsquigarrow \mathrm{tr}(X_i) = 0$
- $(e^{i\alpha_i X_i})^T (e^{i\alpha_j X_j}) = e^{i\alpha_i X_i^T} e^{i\alpha_j X_j} = e^{i\alpha_i (X_i^T + X_i)} \overset{!}{=} \mathbb{1}$

$\Rightarrow X_i^T = -X_i$

$\Rightarrow$ 3 free parameters $\begin{pmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{pmatrix} \rightsquigarrow X_1, X_2, X_3$ as generators for so(3)

For SO(3), or so(3) now?

18 p → 90 % M.M 3 p