Disclaimer

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https://www.physics-and-stuff.com/

I raise no claim to correctness and completeness of the given solutions! This equally applies to the corrections mentioned above.

22.01.2013 Group theory Exercise 11

Marvin Zanke

H11.1 We have 8 Gell-Mann matrices, $\lambda_{i=1..8}$ which fulfill $[\lambda^a, \lambda^b] = 2if^{abc}\lambda^c$

$T^a = \frac{1}{2}\lambda^a \rightsquigarrow Tr[T^aT^b] = \frac{1}{2}\delta^{ab}, \; [T^a, T^b] = if^{abc}T^c$

$$SU(3) = \{U \in SL(3) \mid U^\dagger U = \mathbb{1}, \det U = 1\}$$

So for the underlying Lie Algebra, we demand with

$U = e^{i\alpha_i\lambda_i}$ that $U^\dagger U = \mathbb{1}$ and $\det U = 1$

$\rightsquigarrow$ • $e^{-i\alpha_i\lambda_i^\dagger} e^{i\alpha_i\lambda_i} \overset{!}{=} \mathbb{1} \rightsquigarrow \lambda_i^\dagger = \lambda_i \rightsquigarrow$ hermitian

• $\det(e^{i\alpha_i\lambda_i}) = e^{i\alpha_i tr(\lambda_i)} \overset{!}{=} 1 \rightsquigarrow tr(\lambda_i) = 0 \;\; \forall \lambda_i$

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They also have to be 3x3 matrices, so they obey the form

$$\lambda = \begin{pmatrix} x_1 & y_1 + iy_2 & z_1 + iz_2 \\ y_1 - iy_2 & w_1 & u_1 + iu_2 \\ z_1 - iz_2 & u_1 - iu_2 & -x_1 - w_1 \end{pmatrix}$$

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Other way around? How to show that they are a basis w/o showing their general form?

$\rightsquigarrow \tilde\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; \tilde\lambda_2 = i\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; \tilde\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

$\tilde\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \; \tilde\lambda_5 = i\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$

$\tilde\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \; \tilde\lambda_7 = i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \; \tilde\lambda_8 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

Using $\lambda_i = \tilde\lambda_i$ for $i = 1, 2, 4, 5, 6, 7$

and $\lambda_3 = \tilde\lambda_3 - \tilde\lambda_8$, $\lambda_8 = \frac{1}{\sqrt{3}}(\tilde\lambda_3 + \tilde\lambda_8)$, we find the same matrices as given on the sheet.

+ 2p

- $[\lambda^a, \lambda^b] = 2if^{abc}\lambda^c = -[\lambda_b, \lambda_a] = -2if^{bac}\lambda^c$

$\rightsquigarrow \underline{f^{abc} = -f^{bac}}$

$[\lambda^a, \lambda^b] = 2if^{abc}\lambda^c$

$\Leftrightarrow (\lambda^a\lambda^b - \lambda^b\lambda^a)\lambda^d = 2if^{abc}\lambda^c\lambda^d$

$\Leftrightarrow Tr(\lambda^a\lambda^b\lambda^d - \lambda^b\lambda^a\lambda^d) = 2if^{abc}\, Tr(\lambda^c\lambda^d)$

$\Leftrightarrow Tr([\lambda^b, \lambda^d]\lambda^a) = 2if^{abc}(2\delta^{cd})$

$\Leftrightarrow Tr(2if^{bde}\lambda^e\lambda^a) = 4if^{abc}\delta^{cd}$

$\Leftrightarrow 4if^{bde}\delta^{ea} = 4if^{abd}$

$\rightsquigarrow \underline{f^{bda} = f^{abd} = -f^{bad}}$

$\rightsquigarrow \underline{f^{abc} = -f^{bac} = f^{cab} = -f^{cba}}$

+ 2p

To fulfill

abc	123	147	156	246	257	345	367	458	678
fabc	1	$\underline{1/2}$	$-1/2$	$1/2$	$1/2$	$1/2$	$-1/2$	$\underline{\sqrt{3}/2}$	$\sqrt{3}/2$

We used $[\lambda_1, \lambda_4] = 2if^{14c}\lambda^c$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = i\lambda_7$$

$\rightsquigarrow f^{147} = 1/2$

$[\lambda_4, \lambda_5] = 2if^{45c}\lambda^c$

$$= \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & i \end{pmatrix} = \begin{pmatrix} 2i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2i \end{pmatrix} = 2i\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$= 2i\left\{\frac{1}{2}\left(\sqrt{3}\lambda_8 + \lambda_3\right)\right\} = i\left(\sqrt{3}\lambda_8 + \lambda_3\right)$$

$\rightsquigarrow f^{458} = \sqrt{3}/2$

there exist more str. Const. ?

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2) We look at the matrices T_1, T_2, T_3

What exactly do we have to prove here? Commutator + trace? Also basis?

$$T_1 = \begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \sigma_1/2 & 0 \\ 0 & 0 \end{pmatrix}$$

$$T_2 = \begin{pmatrix} 0 & -i/2 & 0 \\ i/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \sigma_2/2 & 0 \\ 0 & 0 \end{pmatrix}$$

$$T_3 = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \sigma_3/2 & 0 \\ 0 & 0 \end{pmatrix}$$

They obviously fulfill $[T_i, T_j] = [\frac{\sigma_i}{2}, \frac{\sigma_j}{2}] = \frac{i}{2} \epsilon_{ijk} \sigma_k$
$= i \epsilon_{ijk} T_k$, as the Pauli matrices do

$$Tr(T_i T_j) = \frac{1}{4} Tr(\sigma_i \sigma_j) = 1/4 \, Tr(\mathbb{1}_2) \delta_{ij} = 1/2 \, \delta_{ij}$$

For $SU(2)$, we demand $U^\dagger U = \mathbb{1}$, $\det U = 1$

for $U = e^{i\alpha_i \sigma_i}$ $\rightsquigarrow$ $tr(\sigma_i) = 0$ ✓

$\sigma_i^\dagger = \sigma_i$ ✓

and we need $2^2 - 1 = 3$ generators. None of them are linear combinations of each other and thus they form a $SU(2)$ subalgebra.

Why subalg.

3) $SO(3)$: $O^T O = \mathbb{1}$, $\det O = 1$, $O = e^{i\alpha_i X_i}$, X_i: generators

- $(e^{i\alpha_i X_i})^T (e^{i\alpha_j X_j}) \stackrel{!}{=} \mathbb{1} \rightsquigarrow X_i^T + X_i = 0 \rightsquigarrow X_i^T = -X_i$
- $\det(e^{i\alpha_i X_i}) = e^{i\alpha_i tr(X_i)} \stackrel{!}{=} 1 \rightsquigarrow tr(X_i) = 0$

Now look at $T_2 = \begin{pmatrix} 0 & -i/2 & 0 \\ i/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $T_5 = \begin{pmatrix} 0 & 0 & -i/2 \\ 0 & 0 & 0 \\ i/2 & 0 & 0 \end{pmatrix}$

$$T_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i/2 \\ 0 & i/2 & 0 \end{pmatrix}$$

General form: $\begin{pmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{pmatrix}$ $\rightsquigarrow$ 3 free parameters (3 generators)

We need to show closure under this subalgebra and we are done, as we have 3 generators which are no linear comb. of each other and would thus form a basis for the subalgebra.

$$[T_2, T_5] = -\frac{i}{4}\left\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}\right\} = \frac{i}{4}\lambda_7 = \frac{i}{2}T_7$$

Is the table in (1) complete? No other elements? Then easy

$[T_2, T_7] = -\frac{i}{2}T_5$ from table in (1)

$[T_5, T_7] = \frac{i}{2}T_2$ ——— " ———

The relation $Tr[T^a T^b] = \frac{1}{2}\delta^{ab}$ is inherited from $SO(3)$

The matrices are obviously purely imaginary, which means that after absorbing a factor of i or make them real by an isomorphism, they can be taken to be purely real. Hence, it's rather an $SO(3)$ subalgebra, than an $SU(3)$ subalgebra.

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H112

(1) ? Can you send around the solution for this?

Ok.

(2) The master formula reads $\frac{\alpha \cdot \mu}{\alpha^2} = -\frac{1}{2}(p-q)$

applying this to the adjoint repr. ($\mu \hat{=} \alpha, \beta$ roots),

we find

$$\frac{\alpha \cdot \beta}{\alpha^2} = -\frac{1}{2}(p-q)$$

$$\frac{\alpha \cdot \beta}{\beta^2} = -\frac{1}{2}(p'-q')$$

×

multiply $\rightsquigarrow$ $\frac{(\alpha \cdot \beta)^2}{\alpha^2 \beta^2} = \frac{1}{4}(p-q)(p'-q') = \cos^2 \theta_{\alpha\beta}$

divide $\rightsquigarrow$ $\frac{\alpha^2}{\beta^2} = \frac{p'-q'}{p-q}$ with p', q', p, q integer

$\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$, $\cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$, $\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$

$\cos\left(\frac{\pi}{2}\right) = 0$

$\rightsquigarrow$

$4\cos^2\theta_{\alpha\beta}$	$p-q$	$p'-q'$	$\frac{\alpha^2}{\beta^2} = \frac{p'-q'}{p-q}$
3	3 | 1	1 | 3	3 | 1/3
2	2 | 1	1 | 2	2 | 1/2
1	1 | 1	1 | 1	1
0	0 | x	x | 0	not defined

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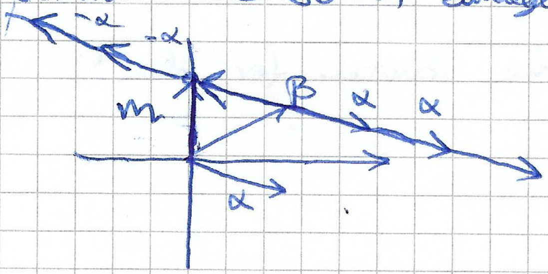
(3) Consider the $SU(2)$ subalgebra spanned by α and β



The states run through the algebra like $\beta + p\alpha$

$\beta - q\alpha$

→ the middle is given by $m = \frac{1}{2}(\beta - q\alpha + \beta + p\alpha)$

$= \beta + \frac{p-q}{2}\alpha$

$\Delta = m - \beta = \frac{p-q}{2}\alpha$

Mirroring the whole subalgebra at m sends every root to its negative and is thus also a solution / a root.

$\beta' = m + \Delta = \beta + \frac{p-q}{2}\alpha + \frac{p-q}{2}\alpha = \beta + (p-q)\alpha$

$= \beta - \frac{2\alpha\cdot\beta}{\alpha^2}\alpha$

↑ Master formula

Formal way?

$p \geq p-q \geq -q$

⇒ a root

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4) $[E_{\alpha_+}, E_{\beta_+}] = 0 \Leftrightarrow (\alpha_+ + \beta_+)$ is not a root

$\neq 0 \Leftrightarrow (\alpha_+ + \beta_+)$ is a root

obviously, $\beta_+ + \alpha_+$ is not a root from fig. (a), thus the commutator vanishes.

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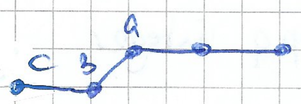
5)



$\alpha_- \beta_- |a\rangle \alpha |c\rangle$

$= \left([\alpha_-, \beta_-] + \beta_- \underbrace{\alpha_-)|a\rangle}_{=0} = [\alpha_-, \beta_-]|a\rangle$

$= 0$ as $[\alpha_-, \beta_-] = 0$ because $\alpha_- + \beta_-$ is no root.

+ 2p

18p → 90% M.M