# Disclaimer

The solution at hand was written in the course of the respective class at the University of Bonn. If not stated differently on top of the first page or the following website, the solution was prepared and handed in solely by me, Marvin Zanke. Anything in a different color than the ball pen blue is usually a correction that I or a tutor made. For more information and all my material, check:
https://www.physics-and-stuff.com/

**I raise no claim to correctness and completeness of the given solutions! This equally applies to the corrections mentioned above.**

21.10.2017 Group theory 2nd Exercise Marvin Zanke

## H2.1

i) Consider the group of permutations of $N$ elements $S_N$.
Take $\bar{P} \in S_N$ which is even. We want to show, that the even permutations $\bar{P}$ form a subgroup $A_N$.

- $e \in A_N$, because the identity is an even permutation:

follows from closure and inverse

$(1) \dots (N) \rightsquigarrow W(e) = (1-1) + \dots (1-1) = 0$

and thus $\delta_e = (-1)^0 = 1$ +

- associativity is inherited from $S_N$ by triviality +
- if $\bar{P} \in A_N$, assume its decomposition into transpositions as follows:

$\bar{P} = (a_1 b_1) \dots (a_m b_m)$, $m$ even

then $\bar{P}^{-1} = (a_m b_m) \dots (a_1 b_1)$ is even as well, and

$$\bar{P}^{-1} P = (a_m b_m) \dots \underbrace{(a_1 b_1)(a_1 b_1)}_{e} \dots (a_m b_m) = e$$

+

obviously and thus $\bar{P}^{-1} \in A_N$.

elements appear twice? $(ab) \dots (ab)$ or $(ab)(bc)$?

- now $\bar{P}_1, \bar{P}_2 \in A_N$, $\bar{P}_1 = (a_1 b_1) \dots (a_m b_m)$, $m$ even,
$\bar{P}_2 = (c_1 d_1) \dots (c_k d_k)$, $k$ even and thus

$$\bar{P}_1 \bar{P}_2 = \underbrace{(a_1 b_1) \dots (a_m b_m)(c_1 d_1) \dots (c_k d_k)}_{m+k \text{ elements}} \quad \text{with } m+k \text{ even}$$

and thus $\bar{P}_1 \bar{P}_2 \in A_N$ +

2p

ii) To derive the order of $A_N$, we consider $A_N = \{\bar{P}_1, \dots, \bar{P}_\ell\}$ with $1 \leq \ell \leq N!$, $\ell$ different elements and the remaining set $S_N \setminus A_N = \{P_1, \dots, P_t\}$ of odd permutations, $t = N! - \ell$

Choose an arbitrary odd permutation $P_i \in S_N \setminus A_N$

necessarily distinct perm.?

$\rightsquigarrow P_i A_N = \{P_i \bar{P}_1, \dots, P_i \bar{P}_\ell\}$ is a set of odd permutations because $P_i = (a_1 b_1) \dots (a_z b_z)$, $z$ odd, $\bar{P}_j = (c_1 d_1) \dots (c_w d_w)$, $w$ even and thus $w+z$ odd and $\ell \leq t$

Multiplying this arbitrary odd permutation $P_i \in S_N \setminus A_N$ with the set of odd permutations $S_N \setminus A_N$, we get:

$P_i \, S_N \setminus A_N = \{P_i P_1, \ldots, P_i P_t\}$, which are even permutations as

$P_i = (a_1 b_1) \ldots (a_z b_z)$, $z$ odd and $P_j = (c_1 d_1) \ldots (c_v d_v)$, $v$ odd

and thus $z + v$ even and $\underline{t \leq l}$.

$\Rightarrow t \leq l$ and $l \leq t \Rightarrow t = l$

$\Rightarrow t = N! - l = N! - t \Leftrightarrow t = N!/2$

$\Rightarrow \underline{l = N!/2}$

3p

What for $N = \sum_j j\alpha_j$?

# #2.2

1) The permutations of N elements (those with the same cycle structure) with $\alpha_j$ = # disjoint cycles of length $\ell$ yield $N = \sum_{j=1}^{k} j\alpha_j$ $(k \leq N)$ for the number of elements.

The number of different permutations with this same cycle structure ($n_{real}$) is given by

$$n_{real} = \frac{N!}{\prod_j j^{\alpha_j} \alpha_j!}.$$

To see this, we look at N elements without any cycle structure for now: $n_1 n_2 \ldots n_N$. The possible ways to commute these N elements amount to $N!$.

Now, we introduce our desired cycles, where we start with the longest cycles from the left, the order of cycles w/ the same length being arbitrary among each other, i.e. +

$$\underbrace{(n_1 \ldots n_j)}_{\text{length } j} \underbrace{(n_{j+1} \ldots n_{2j})}_{\text{length } j} \underbrace{(n_{2j+1} \ldots n_k)}_{\text{length } k-2j} \ldots \underbrace{(n_\ell \ldots n_N)}_{\text{length } N-\ell+1}$$

which here implies $\alpha_j = 2$. As we only fixed the longer cycles to be left of the shorter ones, we can freely interchange the cycles of the same length, as they are disjoint and yield the same cycle structure - permutation, leaving us with $\alpha_j!$ possible ways to do so. As each cycle is cyclic as well, we can ~~permute~~ cycle its elements in a cyclic manner, w/o yielding a different permutation of the same cycle structure, leaving us w/ $j^{\alpha_j}$ ways to do so. Taking into account all cycles, yields a factor of $\prod_j j^{\alpha_j}\alpha_j!$ which has to be divided out of $N!$ permutations.

"cycle permute"?

Everything seems correct, ~~but not convincing.~~ ok. I convinced myself.

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2) [4,0,0,0], [3,1,0,0], [2,2,0,0], [2,1,1,0], [1,1,1,1]
for $S_4$

[4,0,0,0] ≙ [1,1,1,1]
Other way around in lecture?

| Long form | short form | young frames | cycle structure | Parity | in $A_N$ | # elements |
|---|---|---|---|---|---|---|
| [4,0,0,0] | [4] | | (*)(*)(*)(*) | + | ✓ | 1 |
| [3,1,0,0] | [3,1] | | (**)(*)(*) | − | × | 6 |
| [2,2,0,0] | [$2^2$] | | (**)(**) | + | ✓ | 3 |
| [2,1,1,0] | [$2,1^2$] | | (***)(*) | + | ✓ | 8 |
| [1,1,1,1] | [$1^4$] | | (****) | − | × | 6 |

$= \underline{12}$ in $A_N$ $(= \frac{N!}{2})$

+ 5p

A4 part. for N = 4 and the all part. of sym group $S_4$?

3) • $Cl_{[4]} = \{(1)(2)(3)(4)\}$; • $Cl_{[3,1]} = \{(12)(3)(4), (13)(2)(4), (14)(2)(3), (23)(1)(4), (24)(1)(3), (34)(1)(2)\}$

• $Cl_{[2^2]} = \{(12)(34), (13)(24), (14)(23)\}$

• $Cl_{[2,1^2]} = \{(123)(4), (132)(4), (124)(3), (142)(3), (134)(2), (143)(2), (234)(1), (243)(1)\}$

• $Cl_{[1^4]} = \{(1234), (1243), (1423), (1432), (1324), (1342)\}$ +

conjugacy classes of $S_N$ explicitly?

4) [5,0,0,0,0], [4,1,0,0,0], [3,2,0,0,0], [3,1,1,0,0],
[2,2,1,0,0], [2,1,1,1], [1,1,1,1,1] for $S_5$

| long form | short form | young frames | Cycle Structure | Parity | in $A_N$ | # elements |
|---|---|---|---|---|---|---|
| [5,0,0,0,0] | [5] | | (*)(*)(*)(*)(*) | + | ✓ | 1 |
| [4,1,0,0,0] | [4,1] | | (**)(*)(*)(*) | − | × | 10 |
| [3,2,0,0,0] | [3,2] | | (**)(**)(*) | + | ✓ | 15 |
| [3,1,1,0,0] | [$3,1^2$] | | (***)(*)(*) | + | ✓ | 20 |
| [2,2,1,0,0] | [$2^2,1$] | | (***)(**) | − | × | 20 |
| [2,1,1,1,0] | [$2,1^3$] | | (****)(*) | − | × | 30 |
| [1,1,1,1,1] | [$1^5$] | | (*****) | + | ✓ | 24 |

in $A_N$ 60 = $\frac{N!}{2}$

Formular for possibilities of $[x_1, x_2, \_ x_N]$?

Total 20p Maximum 100% Cool
M.M.

+ 5p