Disclaimer

08.11.2017 Group theory Exercise 3 Marvin Zanke

H3.1: $D_4 = \{e, c, c^2, c^3, b, bc, bc^2, bc^3\}$

(1) $= \langle c, b \mid c^4 = e, b^2 = e \rangle$

$(cb)^2 = e \leftarrow$ important +

$\langle c, b \mid \ldots \rangle$ correct?

From Lagrange theorem, we know that any subgroup of D_4 has to be of order $1, 2, 4, 8$.

- order 1 subgroups $H_1 = \{e\}$
- order 8 subgroups $H_8 = D_4$
- order 2 subgroups: cyclic group, isomorphic to $\mathbb{Z}_2$

$bc^i = c^{4-i}b$

All the same group as isomorphic to each other?

thus generated by any element of order 2: c^2, b, bc, bc^2, bc^3

$\Rightarrow H_2^{(1)} = \{e, c^2\}$, $H_2^{(2)} = \{e, b\}$, $H_2^{(3)} = \{e, bc\}$, $H_2^{(4)} = \{e, bc^2\}$

$H_2^{(5)} = \{e, bc^3\}$ +

- order 4 subgroups: either isomorphic to $\mathbb{Z}_4$ ($\Rightarrow$ cyclic) or to $\mathbb{Z}_2 \times \mathbb{Z}_2$, meaning it's generated by an element of order 4 ($= c$) or by two elements of order 2 ($= b, c^2$ or bc, c^2) and thus

$H_4^{(1)} = \{e, c, c^2, c^3\}$, $H_4^{(2)} = \{e, b, c^2, bc^2\}$

$H_4^{(3)} = \{e, bc, c^2, bc^3\}$ +

Conjugacy classes are given by $[a] = \{h \in G : ghg^{-1} = a\}$

- $[e] = \{e\}$

easier way for conjugacy classes? $\Rightarrow$ cycle structure?

No! group S_4 is not complete

- gcg^{-1} where $g = c^n$ or $g = bc^n \Rightarrow$ consider 2 cases
  - $c^n c c^{4-n} = c$
  - $bc^n c c^{4-n} b = bcb = c^3$

$\Rightarrow [c] = \{c, c^3\}$

- gc^2g^{-1}
  - $c^n c^2 c^{4-n} = c^2$
  - $bc^n c^2 c^{4-n} b = bc^2 b = c^2$

$\Rightarrow [c^2] = \{c^2\}$

• gbg^{-1}

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ $n=1,2$ possible

- $c^n b c^{4-n} = c^n c^n b = c^{2n} b$
- $bc^n b c^{4-n} b = bc^n c^n = bc^{2n}$ $\Rightarrow [b] = \{b, bc^2\}$

• $gbcg^{-1}$

- $c^n bc c^{4-n} = bc^{4-n+1} c^{4-n} = bc^{-2n+1}$
- $bc^n bc c^{4-n} b = c^{4-n+1} c^{4-n} b = c^{4-2n+1} b = bc^{2n-1}$ $\Rightarrow [bc] = \{bc, bc^3\}$

(2) Invariant SG means $gH = Hg \ \forall g \in G$ $(H \trianglelefteq G)$

or $ghg^{-1} \in H \ \forall h \in H, \forall g \in G$

No neutral element in those $g\{\dots\}$ objects?

Obviously, invariant SG are the union of conjugacy classes

- $H_1 = \{e\}$ by triviality
- $H_8 = D_4$ by triviality
- $H_2^{(1)} = [e] \cup [c^2] = \{e, c^2\}$
- $H_4^{(1)} = [e] \cup [c] \cup [c^2] = \{e, c, c^2, c^3\}$
- $H_4^{(2)} = [e] \cup [c^2] \cup [b] = \{e, b, c^2, bc^2\}$
- $H_4^{(3)} = [e] \cup [c^2] \cup [bc] = \{e, bc, c^2, bc^3\}$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ fixed

Quotient groups defined by $\{gH \mid H \text{ inv. SG}, g \in G\}$

- $D_4/H_1 = \{\{e\}, \{c\}, \{c^2\}, \{c^3\}, \{b\}, \{bc\}, \{bc^2\}, \{bc^3\}\}$
- $D_4/H_8 = \{H_8\} = \{D_4\}$
- $D_4/H_2^{(1)} = \{\{e, c^2\}, \{c, c^3\}, \{b, bc^2\}, \{bc, bc^3\}\}$
- $D_4/H_4^{(1)} = \{\{e, c, c^2, c^3\}, \{b, bc, bc^2, bc^3\}\}$
- $D_4/H_4^{(2)} = \{\{e, b, c^2, bc^2\}, \{c, bc^3, c^3, bc\}\}$
- $D_4/H_4^{(3)} = \{\{e, bc, c^2, bc^3\}, \{c, b, c^3, bc^2\}\}$

Order not important in a set?

+

How many elements are there in a quotient group?

(3) SG of S_8 that is isomorphic to D_4

construct $\quad g \longmapsto \begin{pmatrix} e & c & c^2 & c^3 & b & bc & bc^2 & bc^3 \\ g & gc & gc^2 & gc^3 & gb & gbc & gbc^2 & gbc^3 \end{pmatrix}$ like in the proof of Cayley's theorem

Consider the 2 cases $g = c^n$ and $g = bc^n$ $\quad$ bijective by rearrangement theorem

Enough like this?!

You need to find the subgroup explicitely.

H3.2

$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_i\sigma_j = \delta_{ij} + i\sum_{k=1}^{3} \epsilon_{ijk}\sigma_k$

We will construct the group table for all elements generated by products of σ_1, σ_2; each new element resulting from a multiplication will be admitted to the table constructively:

	σ_1	σ_2	$\mathbb{1}$	$i\sigma_3$	$-i\sigma_3$	$-\mathbb{1}$	$-\sigma_2$	$-\sigma_1$
σ_1	$\mathbb{1}$	$i\sigma_3$	σ_1	σ_2	$-\sigma_2$	$-\sigma_1$	$-i\sigma_3$	$-\mathbb{1}$
σ_2	$-i\sigma_3$	$\mathbb{1}$	σ_2	$-\sigma_1$	σ_1	$-\sigma_2$	$-\mathbb{1}$	$i\sigma_3$
$\mathbb{1}$	σ_1	σ_2	$\mathbb{1}$	$i\sigma_3$	$-i\sigma_3$	$-\mathbb{1}$	$-\sigma_2$	$-\sigma_1$
$i\sigma_3$	$-\sigma_2$	σ_1	$i\sigma_3$	$-\mathbb{1}$	$\mathbb{1}$	$-i\sigma_3$	$-\sigma_1$	σ_2
$-i\sigma_3$	σ_2	$-\sigma_1$	$-i\sigma_3$	$\mathbb{1}$	$-\mathbb{1}$	$i\sigma_3$	σ_1	$-\sigma_2$
$-\mathbb{1}$	$-\sigma_1$	$-\sigma_2$	$-\mathbb{1}$	$-i\sigma_3$	$i\sigma_3$	$\mathbb{1}$	σ_2	σ_1
$-\sigma_2$	$i\sigma_3$	$-\mathbb{1}$	$-\sigma_2$	σ_1	$-\sigma_1$	σ_2	$\mathbb{1}$	$-i\sigma_3$
$-\sigma_1$	$-\mathbb{1}$	$-i\sigma_3$	$-\sigma_1$	$-\sigma_2$	σ_2	σ_1	$i\sigma_3$	$\mathbb{1}$

- $\sigma_1^2 = \delta_{11} = \mathbb{1}$
- $\sigma_2^2 = \mathbb{1}$
- $\sigma_1\sigma_2 = i\sigma_3$
- $\sigma_2\sigma_1 = -i\sigma_3$
- $\sigma_1 i\sigma_3 = -i^2\sigma_2 = \sigma_2 = -i\sigma_3\sigma_1$
- $\sigma_2 i\sigma_3 = i^2\sigma_1 = -\sigma_1 = -i\sigma_3\sigma_2$
- $i\sigma_3 i\sigma_3 = -\mathbb{1}$

+

- Associativity is inherited by matrix multiplication
- neutral element: $\mathbb{1}$
- inverse: see multiplication table, in each row/column: $\mathbb{1}$

The Group G has the order $|G| = 8$

- For $X = \{\sigma_1, \sigma_2, -\sigma_1, -\sigma_2, -\mathbb{1}\}$, we have $\mathrm{ord}(X) = 2$

  For $Y = \{i\sigma_3, -i\sigma_3\}$, we have $\mathrm{ord}(y) = 4$

  And $\mathrm{ord}(\mathbb{1}) = 1$

We notice, that the inverse of any element is either the element itself, or the negative of the element for $\pm i\sigma_3$. Furthermore, one easy finds, that two different P.M. anticommute, while the same yield the unit matrix (see Levi-Civita symbol). Thus $ghg^{-1} = -h$ for $h \notin \{\mathbb{1}, -\mathbb{1}\}$ and conjugacy classes are disjoint.

Wenn instantly get: $[\mathbb{1}]=\{\mathbb{1}\}$, $[-\mathbb{1}]=\{-\mathbb{1}\}$
and $[\sigma_1]=\{\sigma_1,-\sigma_1\}$, $[\sigma_2]=\{\sigma_2,-\sigma_2\}$, $[i\sigma_3]=\{i\sigma_3,-i\sigma_3\}$
as the negative is in the conjugacy class as well for Pauli-matrices.

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the SG can again be obtained by looking at $|H|=1,2,4,8$ seperately, where $|H|=2$ is generated by an element of order 2 and $|H|=4$ is either generated by an element of order 4 or 2 elements of order 2.

$$\Rightarrow H_1=\{\mathbb{1}\},\ H_8=G,\ H_2^{(1)}=\{\mathbb{1},-\mathbb{1}\},\ H_2^{(2)}=\{\mathbb{1},\sigma_1\}$$
$$H_2^{(3)}=\{\mathbb{1},-\sigma_1\},\ H_2^{(4)}=\{\mathbb{1},\sigma_2\},\ H_2^{(5)}=\{\mathbb{1},-\sigma_2\}$$
$$H_4^{(1)}=\{\mathbb{1},i\sigma_3,-\mathbb{1},-i\sigma_3\},\ H_4^{(2)}=\{\mathbb{1},\sigma_1,-\mathbb{1},-\sigma_1\}$$
$$H_4^{(3)}=\{\mathbb{1},\sigma_2,-\mathbb{1},-\sigma_2\}$$

the invariant SG are: $H_1=[\mathbb{1}]=\{\mathbb{1}\}$, $H_8=G$, $H_2^{(1)}=[\mathbb{1}]\cup[-\mathbb{1}]=\{\mathbb{1},-\mathbb{1}\}$
$$H_4^{(1)}=[\mathbb{1}]\cup[-\mathbb{1}]\cup[i\sigma_3]=\{\mathbb{1},i\sigma_3,-\mathbb{1},-i\sigma_3\}$$
$$H_4^{(2)}=[\mathbb{1}]\cup[-\mathbb{1}]\cup[\sigma_1]=\{\mathbb{1},\sigma_1,-\mathbb{1},-\sigma_1\}$$
$$H_4^{(3)}=[\mathbb{1}]\cup[-\mathbb{1}]\cup[\sigma_2]=\{\mathbb{1},\sigma_2,-\mathbb{1},-\sigma_2\}$$

Now construct the Quotient groups $\{gH\}$, where $|g_iH|=|eH|$ and $\bigcup gH=G$

$$G/H_1=\{\{\mathbb{1}\},\{\sigma_1\},\{\sigma_2\},\{i\sigma_3\},\{-i\sigma_3\},\{-\sigma_1\},\{-\sigma_2\},\{-\mathbb{1}\}\}$$
$$G/H_8=\{H_8\}=\{G\}$$
$$G/H_2^{(1)}=\{\{\mathbb{1},-\mathbb{1}\},\{\sigma_1,-\sigma_1\},\{\sigma_2,-\sigma_2\},\{i\sigma_3,-i\sigma_3\}\}$$
$$G/H_4^{(1)}=\{\{\mathbb{1},i\sigma_3,-\mathbb{1},-i\sigma_3\},\{\sigma_1,-\sigma_1,\sigma_2,-\sigma_2\}\}$$
$$G/H_4^{(2)}=\{\{\mathbb{1},\sigma_1,-\mathbb{1},-\sigma_1\},\{\sigma_2,-\sigma_2,i\sigma_3,-i\sigma_3\}\}$$
$$G/H_4^{(3)}=\{\{\mathbb{1},\sigma_2,-\mathbb{1},-\sigma_2\},\{\sigma_1,-\sigma_1,i\sigma_3,-i\sigma_3\}\}$$

Construct isomorphism: $\varphi: D_4\to G$ and $\varphi(b)=\sigma_1$, $\varphi(c)=i\sigma_3$, $\varphi(e)=\mathbb{1}$ for the generators. then define $\varphi(b^n c^m)=\varphi(b^n)\varphi(c^m)=\varphi(b)^n\varphi(c)^m$.
We have $\mathrm{ord}(\varphi(b))=\mathrm{ord}(b)$, $\mathrm{ord}(\varphi(c))=\mathrm{ord}(c)$

$\Rightarrow \varphi(c^2)=-\mathbb{1}$, $\varphi(c^3)=-i\sigma_3$, $\varphi(bc)=\sigma_2$, $\varphi(bc^2)=-\sigma_1$, $\varphi(bc^3)=-\sigma_2$

WHY NOT POSSIBLE LIKE THIS?

H3.2

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_i \sigma_j = \delta_{ij} + i \sum_{k=1}^{3} \epsilon_{ijk} \sigma_k$$

We will construct the group table for all elements generated by products of σ_1, σ_2; each new element resulting from a multiplication will be admitted to the table constructively:

see that it forms a group by the table? e.g. inverse element otherwise not easy to see?

	σ_1	σ_2	$\mathbb{1}$	$i\sigma_2$	$-i\sigma_3$	$-\mathbb{1}$	$-\sigma_2$	$-\sigma_1$
σ_1	$\mathbb{1}$	$i\sigma_3$	σ_1	σ_2	$-\sigma_2$	$-\sigma_1$	$-i\sigma_3$	$-\mathbb{1}$
σ_2	$-i\sigma_3$	$\mathbb{1}$	σ_2	$-\sigma_1$	σ_1	$-\sigma_2$	$-\mathbb{1}$	$i\sigma_3$
$\mathbb{1}$	σ_1	σ_2	$\mathbb{1}$	$i\sigma_3$	$-i\sigma_3$	$-\mathbb{1}$	$-\sigma_2$	$-\sigma_1$
$i\sigma_3$	$-\sigma_2$	σ_1	$i\sigma_3$	$-\mathbb{1}$	$\mathbb{1}$	$-i\sigma_3$	$-\sigma_1$	σ_2
$-i\sigma_3$	σ_2	$-\sigma_1$	$-i\sigma_3$	$\mathbb{1}$	$-\mathbb{1}$	$i\sigma_3$	σ_1	$-\sigma_2$
$-\mathbb{1}$	$-\sigma_1$	$-\sigma_2$	$-\mathbb{1}$	$-i\sigma_3$	$i\sigma_3$	$\mathbb{1}$	σ_2	σ_1
$-\sigma_2$	$i\sigma_3$	$-\mathbb{1}$	$-\sigma_2$	σ_1	$-\sigma_1$	σ_2	$\mathbb{1}$	$-i\sigma_3$
$-\sigma_1$	$-\mathbb{1}$	$-i\sigma_3$	$-\sigma_1$	$-\sigma_2$	σ_2	σ_1	$i\sigma_3$	$\mathbb{1}$

- $\sigma_1^2 = \delta_{11} = \mathbb{1}$
- $\sigma_2^2 = \mathbb{1}$
- $\sigma_1 \sigma_2 = i\sigma_3$
- $\sigma_2 \sigma_1 = -i\sigma_3$
- $\sigma_1 i\sigma_3 = -i^2 \sigma_2 = \sigma_2 = -i\sigma_3\sigma_1$
- $\sigma_2 i\sigma_3 = i^2 \sigma_1 = -\sigma_1 = -i\sigma_3\sigma_2$
- $i\sigma_3 i\sigma_3 = -\mathbb{1}$

Yes! you can if it is consistent: every element is only once in row and column

- Associativity is inherited by matrix multiplication
- neutral element: $\mathbb{1}$
- inverse: See multiplication table, in each row/column: $\mathbb{1}$

- The Group G has the order $|G| = 8$
- For $X = \{\sigma_1, \sigma_2, -\sigma_1, -\sigma_2, -\mathbb{1}\}$, we have $\mathrm{ord}(X) = 2$

  For $Y = \{i\sigma_3, -i\sigma_3\}$, we have $\mathrm{ord}(Y) = 4$

  And $\mathrm{ord}(\mathbb{1}) = 1$

Why not use the isomorphism to get the classes etc faster? Otherwise twice the work?

We now construct a trivial isomorphism $f: D_4 \to G$:

$f(b) = \sigma_1, \; f(bc^3) = \sigma_2, \; f(e) = \mathbb{1}, \; f(c^3) = i\sigma_3$

$f(c) = -i\sigma_3, \; f(c^2) = -\mathbb{1}, \; f(bc) = -\sigma_2, \; f(bc^2) = -\sigma_1$

Then $f(X) = \{f(x) \mid x \in X\}$ obviously fulfills $\mathrm{ord}(f(X)) = \mathrm{ord}(X)$

$f(Y) = \{f(y) \mid y \in Y\}$ fulfills $\mathrm{ord}(f(Y)) = \mathrm{ord}(Y)$

$f(e) = \mathbb{1} \Rightarrow \mathrm{ord}(f(e)) = \mathrm{ord}(\mathbb{1})$

· $f(a)f(b) = f(ab)$ can be checked w/ the multiplication tables

· By construction it's injective and because of equal sizes surjective

The conjugacy classes can thus be taken from H3.1 as:

· $[\mathbb{1}] = \{\mathbb{1}\}$, $[\sigma_1] = \{\sigma_1, -\sigma_1\}$, $[\sigma_2] = \{\sigma_2, -\sigma_2\}$
$[i\sigma_3] = \{i\sigma_3, -i\sigma_3\}$, $[-\mathbb{1}] = \{-\mathbb{1}\}$

The SG are: $H_1 = \{\mathbb{1}\}$, $H_8 = G$, $H_2^{(1)} = \{\mathbb{1}, -\mathbb{1}\}$, $H_2^{(2)} = \{\mathbb{1}, \sigma_1\}$
$H_2^{(3)} = \{\mathbb{1}, -\sigma_2\}$, $H_2^{(4)} = \{\mathbb{1}, -\sigma_1\}$, $H_2^{(5)} = \{\mathbb{1}, \sigma_2\}$
$H_4^{(1)} = \{\mathbb{1}, -i\sigma_3, -\mathbb{1}, i\sigma_3\}$, $H_4^{(2)} = \{\mathbb{1}, \sigma_1, -\mathbb{1}, -\sigma_1\}$
$H_4^{(3)} = \{\mathbb{1}, -\sigma_2, -\mathbb{1}, \sigma_2\}$

Invariant of these are: $H_1 = \{e\}$, $H_8 = G$, $H_2^{(1)} = [\mathbb{1}] \cup [-\mathbb{1}] = \{\mathbb{1}, -\mathbb{1}\}$
$H_4^{(1)} = [\mathbb{1}] \cup [i\sigma_3] \cup [-\mathbb{1}] = \{\mathbb{1}, -i\sigma_3, -\mathbb{1}, i\sigma_3\}$
$H_4^{(2)} = [\mathbb{1}] \cup [-\mathbb{1}] \cup [\sigma_1] = \{\mathbb{1}, \sigma_1, -\mathbb{1}, -\sigma_1\}$
$H_4^{(3)} = [\mathbb{1}] \cup [-\mathbb{1}] \cup [\sigma_2] = \{\mathbb{1}, \sigma_2, -\mathbb{1}, -\sigma_2\}$

We now construct the quotient groups:

$G/H_1 = \{\{\mathbb{1}\}, \{-i\sigma_3\}, \{-\mathbb{1}\}, \{i\sigma_3\}, \{\sigma_1\}, \{-\sigma_2\}, \{-\sigma_1\}, \{\sigma_2\}\}$
$G/H_8 = \{H_8\} = \{G\}$
$G/H_2^{(1)} = \{\{\mathbb{1}, -\mathbb{1}\}, \{-i\sigma_3, i\sigma_3\}, \{\sigma_1, -\sigma_1\}, \{-\sigma_2, \sigma_2\}\}$
$G/H_4^{(1)} = \{\{\mathbb{1}, -i\sigma_3, -\mathbb{1}, i\sigma_3\}, \{\sigma_1, -\sigma_2, -\sigma_1, \sigma_2\}\}$
$G/H_4^{(2)} = \{\{\mathbb{1}, \sigma_1, -\mathbb{1}, -\sigma_1\}, \{-i\sigma_3, \sigma_2, i\sigma_3, -\sigma_2\}\}$
$G/H_4^{(3)} = \{\{\mathbb{1}, -\sigma_2, -\mathbb{1}, \sigma_2\}, \{-i\sigma_3, \sigma_1, i\sigma_3, -\sigma_1\}\}$

?

Is it ok to first show the isomorphism to D_4 and then just copy the properties? Or why should it be shown last that $G \simeq D_4$? Otherwise really hard to find conjugacy classes etc.?

I think yes It is ok either ways

20/20 → 100%