Disclaimer

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Group theory Exercise 3

H3.1: \( D_4 = \{ e, c, c^2, c^3, b, bc, bc^2, bc^3 \} \)

(1) \( \langle c, b \mid c^4 = e, b^2 = c, (cb)^2 = e \rangle \) is important!

From Lagrange's theorem, we know that any subgroup of \( D_4 \) has to be of order 1, 2, 4, 8.

- Order 1 subgroups \( H_1 = \{ e \} \)
- Order 2 subgroups \( H_2 = \{ e, c^2 \} \)
- Order 4 subgroups isomorphic to \( \mathbb{Z}_2 \)
  
  thus generated by any element of order 2: \( c, b, bc, bc^2, bc^3 \)

\( H_2 = \{ e, c^2 \} \), \( H_2 = \{ e, b \} \), \( H_2 = \{ e, bc \} \), \( H_2 = \{ e, bc^2 \} \)

Thus, \( H_2 = \{ e, bc \} \)

- Order 4 subgroups isomorphic to \( \mathbb{Z}_4 \) (cyclic) or to \( \mathbb{Z}_2 \times \mathbb{Z}_2 \), meaning it's generated by an element of order 4 (= c) or by two elements of order 2 \(( b, c^2 \) or \( bc, c^2 \)) and thus

\( H_2 = \{ e, c, c^2, c^3 \} \), \( H_2 = \{ e, b, c^2, bc^2 \} \)

Conjugacy classes are given by \( [a] = \{ \text{all } g \in G : gag^{-1} = a \} \)

- \( [e] = \{ e \} \)

\( g \notin \mathbb{Z}_4 \) where \( g = c^2 \) or \( g = bc^2 \) \( \Rightarrow \) consider 2 cases

- \( c^2 c^{4-n} = c \)

- \( bc^2 c^{4-n} b = bc^2 b = c^2 \)

- \( bc^2 c^{4-n} b = bc^2 b = c^2 \)
9 b c^{-1}
- e^n b c^{4-n} = c^n b = c b^{2n} \implies \{b\} = \{b, bc^2\}

9 b c^{-1}
- c b c^{4-n} = b c c^{-n} = b c^{2n} \implies \{bc\} = \{bc, bc^2\}

2) Invariant SGA means \( g H = Hg \forall g \in G \) (it SGA)

or \( g H g^{-1} \in H \forall g \in G \)

Obviously, Invariant SGA are the union of conjugacy classes

- \( H_e = \{e\} \) by triviality
- \( H_e = D_4 \) by triviality,
- \( H_{e_1} = \{e\} \cup \{c^2\} = \{e, c^2\} \)
- \( H_{e_2} = \{e\} \cup \{c\} \cup \{c^2\} = \{e, c, c^2\} \)
- \( H_{e_3} = \{e\} \cup \{c^2\} \cup \{b\} = \{e, b, c^2, bc^2\} \)
- \( H_{e_4} = \{e\} \cup \{c^2\} \cup \{bc\} = \{e, bc, c^2, bc^2\} \)

Quotient groups defined by \( \{ g H \mid H \text{ w.s. } g \in G \} \)

- \( D_4 / H_1 = \{ e, b, c, c^2, c^3, bc, bc^2, bc^3 \} \)
- \( D_4 / H_2 = \{ H_2 \} = \{ D_4 \} \)
- \( D_4 / H_3 = \{ e, c, c^3, b, bc^2, bc \} \)
- \( D_4 / H_4 = \{ e, bc, c^2, bc^2, c^3, bc \} \)
- \( D_4 / H_5 = \{ e, bc, c^2, bc^2, c^3, bc \} \)
- \( D_4 / H_6 = \{ e, bc, c^2, bc^2 \} \)

(2) SGs of SGA had isomorphic to D4

Correct \( g \mapsto (e, c, c^2, c^3, b, bc, bc^2, bc^3) \) take as the proof of Cayley's theorem

Consider the 2 cases \( g = c^n \) and \( g = bc^n \) objective by rearrange-

you need to find the subgroup explicitly.
\[ H3.2 \]

\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_i \sigma_j = \delta_{ij} + \frac{3}{2} \epsilon_{ijk} \sigma_k \]

We will construct the group table for all elements generated by products of \( \sigma_1, \sigma_2 \); each new element resulting from a multiplication will be added to the table constructively.

<table>
<thead>
<tr>
<th>( \sigma_1 )</th>
<th>( \sigma_2 )</th>
<th>( \sigma_1 \sigma_2 )</th>
<th>( \sigma_2 \sigma_1 )</th>
<th>( -\sigma_2 )</th>
<th>( -\sigma_1 )</th>
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</tr>
</tbody>
</table>

- \( \sigma_1^2 = \delta_{11} \quad -\sigma_2^2 = \delta_{22} \)
- \( \sigma_1 \sigma_2 = i \sigma_3 
- \sigma_2 \sigma_1 = -i \sigma_3 
- \sigma_1^2 - \sigma_2^2 = \sigma_1 \sigma_2 = i \sigma_3 
- \sigma_2^2 - \sigma_1^2 = \sigma_2 \sigma_1 = -i \sigma_3 
- \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 = \sigma_3 
- \sigma_2 \sigma_1 \sigma_2 = \sigma_1 \sigma_2 \sigma_1 = i \sigma_3 
- \sigma_1 \sigma_2 \sigma_1 \sigma_2 = \sigma_2 \sigma_1 \sigma_2 \sigma_1 = i \sigma_3 
- \sigma_2 \sigma_1 \sigma_2 \sigma_1 = \sigma_1 \sigma_2 \sigma_1 \sigma_2 = -i \sigma_3 
- \sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \sigma_1 \sigma_2 = -i \sigma_3 
- \sigma_2 \sigma_1 \sigma_2 \sigma_1 \sigma_2 = \sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_1 = i \sigma_3 

- Associativity is inherited by matrix multiplication
- Neutral element: \( I \)
- Inverse: See multiplicative inverse, \( \sigma_i^{-1} = \sigma_i \) in each row/column \( I \)

The group \( G \) has the order \( 16 \times 1 = 16 \)

For \( X = \sigma_1, \sigma_2, -\sigma_1, -\sigma_2, -I \), we have \( \text{ord}(X) = 2 \)

For \( Y = 3 \sigma_3, -i \sigma_3 \), we have \( \text{ord}(Y) = 4 \)

And \( \text{ord}(\sigma_3 \sigma_1) = 1 \)

We notice that the inverse of any element is either the element itself, or the negative of the element for \( \pm i \sigma_3 \).

Furthermore, one easily finds that two different \( P \times I \), anticommute, while the same yield the unit matrix (see Levi-Civita symbol).

Thus \( G \sigma_1 G^{-1} = -I \) for \( G \in \{ i \sigma_3, -i \sigma_3 \} \) and adjacency classes are disjoint.
We immediately get: \[ [A] = \{-1\}, [-A] = -1 \]
and \[ [i] = i, [1] = 1 \]
as the negative is in the conjugacy class as well for Pauli-matrices.

The S_3 can again be obtained by looking at \( \{H\} = \{1, 2, 4, 8\} \), respectively, where \( \{H\} = \{1\} \) is generated by an element of order 2 and \( \{H\} = \{1, 2\} \) is either generated by an element of order 4 or 2 elements of order 2.

- \( H_1 = \{1\} \)
- \( H_2 = \{1, -1\} \)
- \( H_3 = \{1, i, -1, -i\} \)
- \( H_4 = \{1, i, -1, 1, i\} \)

The invariant S_3 are: \( H_4 = \{1\} \)

\[ H_5 = \{1\} \cup \{A\} \cup \{1, i\} \]
\[ H_6 = \{1\} \cup \{A\} \cup \{1, -i\} \]
\[ H_7 = \{1\} \cup \{A\} \cup \{i, -1\} \]
\[ H_8 = \{1\} \cup \{A\} \cup \{-1, -i\} \]

Now consider the quotient groups \( \{G/H\} \) where \( \{1, H\} = \text{left} \) and \( \text{right} \) gives:

\[ G/H_1 = \{ A, -A, i, -i, 1, -1, iA, -iA, i, -i, 1, -1 \} \]
\[ G/H_2 = \{ 1, -1 \} \]
\[ G/H_3 = \{ A, -A, i, -i, 1, -1, i, -i, 1, -1 \} \]
\[ G/H_4 = \{ 1, -1 \} \]
\[ G/H_5 = \{ A, -A \} \]
\[ G/H_6 = \{ A, -A, i, -i \} \]
\[ G/H_7 = \{ A, -A, i, -i \} \]
\[ G/H_8 = \{ A, -A \} \]

Construct isomorphism \( \phi : D_4 \to G \) and \( \phi(1) = 1, \phi(c) = i, \phi(c^2) = 1, \phi(c^3) = 1 \).

We have \( \text{ord}(c), \text{ord}(bc), \text{ord}(bc^2), \text{ord}(bc^3), \text{ord}(bc^4) \).

\[ \phi(c^4) = A, \phi(c^5) = -1 \]
\[ \phi(c^6) = -i, \phi(c^7) = i, \phi(c^8) = 1 \]
Why not possible like this?

\[ H_3.2 \quad \sigma_1 = (0 1), \quad \sigma_2 = (1 0) \quad \sigma_i \sigma_j = \delta_{ij} + \sum_{k=1}^{2} E_{ijk} \sigma_k \]

We will construct the group table for all elements generated by products of \( \sigma_1, \sigma_2 \); each new element resulting from a multiplication will be admitted to the table constructively.

\[
\begin{array}{cccccccc}
\sigma_1 & \sigma_2 & 1 & i \sigma_2 & -i \sigma_2 & -1 & -\sigma_2 & -1 \\
\sigma_1 & 1 & i \sigma_2 & -\sigma_1 & -i \sigma_2 & \sigma_1 & \sigma_2 & \sigma_1 \\
i \sigma_2 & i \sigma_2 & 1 & -i \sigma_2 & \sigma_1 & -\sigma_1 & \sigma_2 & -\sigma_1 \\
\sigma_1 & -\sigma_1 & -i \sigma_2 & 1 & i \sigma_2 & \sigma_1 & \sigma_2 & \sigma_1 \\
-\sigma_1 & -i \sigma_2 & -1 & \sigma_1 & 1 & -i \sigma_2 & \sigma_1 & -i \sigma_2 \\
\sigma_1 & -\sigma_1 & -\sigma_2 & -1 & \sigma_2 & 1 & i \sigma_2 & \sigma_1 \\
-\sigma_1 & i \sigma_2 & 1 & \sigma_1 & -i \sigma_2 & \sigma_1 & 1 & -i \sigma_2 \\
\end{array}
\]

Yes! You can if it is consistent; every element exists once in row and column.

- Associativity is inherited by matrix multiplication.
- Neutral element: 1.
- Inverse: See multiplication table, in each row column: 1.

- The group \( G \) has the order \( |G| = 8 \).
- For \( X = \{\sigma_1, i \sigma_2, -\sigma_1, -i \sigma_2\} \), we have \( \text{ord}(X) = 2 \).
- For \( Y = \{i \sigma_2, -i \sigma_2\} \), we have \( \text{ord}(Y) = 4 \).
- And \( \text{ord}(A) = 1 \).

We now construct a trivial isomorphism \( \varphi : D_4 \to G_6 \).

\[ \begin{align*}
\varphi(\sigma_1) &= \sigma_1, \quad \varphi((bc)^2) = \sigma_2, \quad \varphi(i) = 1, \quad \varphi(c^2) = i \sigma_2 \\
\varphi(c) &= i \sigma_2, \quad \varphi((c^2)^2) = -1, \quad \varphi(bc) = -\sigma_2, \quad \varphi((bc)^2) = -\sigma_1 \\
\end{align*} \]

Then \( \varphi(\sigma_1) \cdot \varphi(\sigma_2) = \varphi(i \sigma_2) \cdot \varphi(1) = \varphi(1) \cdot \varphi(1) \), obviously fulfill \( \text{ord}(\varphi(\sigma_1)) = \text{ord}(\sigma_1) \).

\[ \varphi(\sigma_1) = 1 \quad \text{and} \quad \varphi(\sigma_2) = \text{ord}(\varphi(1)) = \text{ord}(1) \]
\[ P(a) + P(b) = P(ab) \text{ can be checked with the multiplication tables.} \]

By construction, it is injective and because of equal sizes surjective.

The conjugacy classes can thus be taken from Table 1, as:

- \([1] = \{ e \} = \{ \sigma_1 \}, \quad [\sigma_1] = \{ \sigma_1, -\sigma_1 \} \]
- \([\sigma_2] = \{ \sigma_2, -\sigma_2 \} \]
- \([i_{\sigma_2}] = \{ i_{\sigma_2}, -i_{\sigma_2} \} \]

The SGA are:

- \(H_1 = [M], \quad H_2 = G, \quad H_2^{(2)} = [M], \quad H_2^{(3)} = [M, \sigma_1]\)
- \(H_2^{(4)} = \{ M, -\sigma_1 \}, \quad H_2^{(5)} = \{ M, \sigma_1, -\sigma_1 \}, \quad H_2^{(6)} = \{ M, \sigma_1, -\sigma_1, -\sigma_2 \}, \quad H_2^{(7)} = \{ M, -\sigma_2, -\sigma_2 \} \)

Invariant of these are:

- \(H_1 = \{ e \} \)
- \(H_2 = G \)
- \(H_2^{(2)} = [M] \cup [\sigma_1] = [M, \sigma_1] \)
- \(H_2^{(3)} = [M] \cup [\sigma_3] \cup [-\sigma_3] = \{ M, -\sigma_3, M, i_{\sigma_3} \} \)
- \(H_2^{(4)} = [M] \cup [\sigma_4] \cup [-\sigma_4] = \{ M, \sigma_4, -\sigma_4 \}, \quad H_2^{(5)} = \{ M, \sigma_4, -\sigma_4, -\sigma_1 \} \)
- \(H_2^{(6)} = [M] \cup [-\sigma_2] \cup [\sigma_2] = \{ M, \sigma_2, -M, -\sigma_2 \} \)

We now construct the quotient groups:

- \(G/H_1 = \{ M, M, -i_{\sigma_3}, -i_{\sigma_3}, i_{\sigma_3}, i_{\sigma_3}, -\sigma_1, -\sigma_1, i_{\sigma_2}, i_{\sigma_2} \} \)
- \(G/H_2 = \{ G \} \)
- \(G/H_2^{(2)} = \{ M, M, -i_{\sigma_3}, i_{\sigma_3}, -i_{\sigma_3}, i_{\sigma_3}, -\sigma_1, -\sigma_1, i_{\sigma_2}, i_{\sigma_2} \} \)
- \(G/H_2^{(3)} = \{ M, M, -i_{\sigma_3}, i_{\sigma_3}, -i_{\sigma_3}, i_{\sigma_3}, -\sigma_1, -\sigma_1, i_{\sigma_2}, i_{\sigma_2} \} \)
- \(G/H_2^{(4)} = \{ M, i_{\sigma_4}, -M, -i_{\sigma_4}, i_{\sigma_4}, -\sigma_1, -\sigma_1, -\sigma_2, \sigma_2 \} \)
- \(G/H_2^{(5)} = \{ M, i_{\sigma_4}, -M, -i_{\sigma_4}, i_{\sigma_4}, -\sigma_1, -\sigma_1, -\sigma_2, \sigma_2 \} \)
- \(G/H_2^{(6)} = \{ M, -\sigma_2, -M, \sigma_2, i_{\sigma_2}, i_{\sigma_2}, -\sigma_3, \sigma_3 \} \)

Is it ok to first show the isomorphism to \(D_4\) and then just copy the properties? Or why should it be shown last that \(G = D_4\)? Otherwise really hard to find conjugacy classes etc.?

I think yes, it is ok either ways.

\[ 20p \rightarrow 100p \]