

Disclaimer

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<https://www.physics-and-stuff.com/>

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H4.1

$[3, 2, 1^3]$

Y.F.



Cycle structure

$(*) (*) (*) (*) (*) (*) (*) (*) (*)$
 even odd even

Parity

-1

elements

$\frac{8!}{5 \cdot 2 \cdot 2} = 4032$

+ 3p

$[3^2, 1^2]$



$(*) (*) (*) (*) (*) (*) (*) (*) (*)$

-1

$\frac{8!}{4 \cdot 2^2 \cdot 2!} = 1760$

+ 3p

H4.2

(1) $Q_4 = \langle i\sigma_1, i\sigma_2 \mid \sigma_i \sigma_j = \delta_{ij} + i \epsilon_{ijk} \sigma_k \rangle$

How to get order before constructing table?

	$i\sigma_1$	$i\sigma_2$	-1	$i\sigma_3$	$-i\sigma_3$	1	$-i\sigma_2$	$-i\sigma_1$
$i\sigma_1$	-1	$-i\sigma_3$	$-i\sigma_1$	$i\sigma_2$	$-i\sigma_2$	$i\sigma_1$	$i\sigma_3$	1
$i\sigma_2$	$i\sigma_3$	-1	$-i\sigma_2$	$-i\sigma_1$	$i\sigma_1$	$i\sigma_2$	1	$-i\sigma_3$
-1	$-i\sigma_1$	$-i\sigma_2$	1	$-i\sigma_3$	$i\sigma_3$	-1	$i\sigma_2$	$i\sigma_1$
$i\sigma_3$	$-i\sigma_2$	$i\sigma_1$	$-i\sigma_3$	1	1	$i\sigma_3$	$-i\sigma_1$	$i\sigma_2$
$-i\sigma_3$	$i\sigma_2$	$-i\sigma_1$	$i\sigma_3$	1	-1	$-i\sigma_3$	$i\sigma_1$	$-i\sigma_2$
1	$i\sigma_1$	$i\sigma_2$	-1	$i\sigma_3$	$-i\sigma_3$	1	$-i\sigma_2$	$-i\sigma_1$
$-i\sigma_2$	$-i\sigma_3$	1	$i\sigma_2$	$i\sigma_1$	$-i\sigma_1$	$-i\sigma_2$	-1	$i\sigma_3$
$-i\sigma_1$	1	$i\sigma_3$	$i\sigma_1$	$-i\sigma_2$	$i\sigma_2$	$-i\sigma_1$	$-i\sigma_3$	-1

$(i\sigma_1)^2 = -1$

$i\sigma_1 i\sigma_2 = -i\sigma_3$

$i\sigma_2 i\sigma_1$ using $\{\sigma_i, \sigma_j\} = 0$

from $\sigma_i \sigma_j = \delta_{ij} + i \epsilon_{ijk} \sigma_k$

$i\sigma_1 i\sigma_3 = -(-i\sigma_2) = i\sigma_2$

$i\sigma_2 i\sigma_3 = -i\sigma_1$

$|Q_4| = 8$

+ 3p

$\text{ord}(1) = 1, \text{ord}(-1) = 2$

$\text{ord}(X) = 4, X = \{i\sigma_1, i\sigma_2, i\sigma_3, -i\sigma_3, -i\sigma_2, -i\sigma_1\}$

(2) Conjugacy classes: $gHg^{-1} = H$ or $ghg^{-1} \in H \forall g \in Q_4, h \in H$

$[1] = \{1\}$ and $[-1] = \{-1\}$ trivial

General case for the remaining elements: $(\pm i\sigma_j)(\pm i\sigma_i)(\mp i\sigma_j)$

as $g \in \{1, -1\}$ trivial as well. $\Rightarrow \sigma_j (\pm i\sigma_i) \sigma_j = \sigma_j^2 (\mp i\sigma_i) = \mp i\sigma_i$

$\Rightarrow [i\sigma_1] = \{i\sigma_1, -i\sigma_1\}, [i\sigma_2] = \{i\sigma_2, -i\sigma_2\}, [i\sigma_3] = \{i\sigma_3, -i\sigma_3\}$

+ 3p

(3) Following from Lagrange's theorem, all SGs have to be of orders 1, 2, 4, 8

$H_1 = \{1\}$, $H_8 = Q_8$; $H_2^{(1)} = \{1, -1\}$ as it has to be generated by an element of order 2. All other SGs can only be generated by an element of order 4 (see last page) and thus $\cong C_4$:

$$H_4^{(1)} = \{1, i\sigma_1, -1, -i\sigma_1\}, H_4^{(2)} = \{1, i\sigma_2, -1, -i\sigma_2\}, H_4^{(3)} = \{1, i\sigma_3, -1, -i\sigma_3\}$$

Proper invariant SGs ($\neq H_1, H_8$) have to be the union of conjugacy classes:

$$H_2^{(1)} = [1] \cup [-1], H_4^{(1)} = [1] \cup [-1] \cup [i\sigma_1],$$

$$H_4^{(2)} = [1] \cup [-1] \cup [i\sigma_2], H_4^{(3)} = [1] \cup [-1] \cup [i\sigma_3] \quad + 3p$$

How to see if we have all SG?

(4) $Q_8/H_1 = \{gH \mid g \in Q_8\}$

$$Q_8/H_2 = \{\{i\sigma_1\}, \{i\sigma_2\}, \{-1\}, \{i\sigma_3\}, \{-i\sigma_3\}, \{1\}, \{-i\sigma_2\}, \{-i\sigma_1\}\}$$

$$Q_8/H_8 = Q_8 \quad \text{Identify groups } Z_2, Z_4, U?$$

$$Q_8/H_4^{(1)} = \{\{i\sigma_1, -i\sigma_1\}, \{i\sigma_2, -i\sigma_2\}, \{i\sigma_3, -i\sigma_3\}, \{1, -1\}\}$$

$$Q_8/H_4^{(2)} = \{\{1, i\sigma_1, -1, -i\sigma_1\}, \{i\sigma_2, i\sigma_3, -i\sigma_2, -i\sigma_3\}\}$$

$$Q_8/H_4^{(3)} = \{\{1, i\sigma_2, -1, -i\sigma_2\}, \{i\sigma_1, -i\sigma_3, -i\sigma_1, i\sigma_3\}\}$$

$$Q_8/H_4^{(1)} = \{\{1, i\sigma_3, -1, -i\sigma_3\}, \{i\sigma_1, i\sigma_2, -i\sigma_1, -i\sigma_2\}\} \quad \text{Ip}$$

Quotient groups also for non-proper?

(5) $D_4 = \langle a, b \mid a^4 = e, b^2 = e, (bc)^2 = e \rangle$

$$(bc)^2 = e = bc bc \Leftrightarrow bcb = c^{-1} \Leftrightarrow bcb^{-1} = c^{-1} \quad \text{Ip}$$

(6) D_4 has a generator of order 4 and a generator of order 2.

So -1 is the only choice for the generator of order 2.

Then (5) has to be fulfilled for the other generator for an isomorphism:

Why does this have to hold if we want an isomorphism?

$$(-1)c(-1) \stackrel{!}{=} c^{-1} \Leftrightarrow c = c^{-1} \Leftrightarrow c^2 = e$$

But we don't have another element of order 2. \rightarrow not possible

$$Q_8 = \langle a = i\sigma_2, b = i\sigma_1 \mid a^4 = 1, b^2 = -1, bab^{-1} = a^{-1} = -a \rangle$$

$$\text{as } (i\sigma_1)^2 = -1, (i\sigma_2)^2 = -1$$

$$(i\sigma_2)(i\sigma_1)(i\sigma_2)^{-1} = i\sigma_2 \sigma_1 \sigma_2 = -i\sigma_1 \quad + 2p$$

IS \rightarrow 95% H.M.