Disclaimer

15.11.2017 Group Theory Exercise 4

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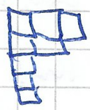
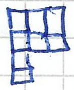
H4.1

	Y.F.	Cycle structure	Parity	# elements
$[3,2,1^3]$ $\hat{=}$	 $\hat{=}$	$\underbrace{(*****)}_{\text{even}}\underbrace{(**)}_{\text{odd}}\underbrace{(*)}_{\text{even}}$	-1	$\frac{8!}{5\cdot 2\cdot 1} = 4032$
$[3^2, 1^2]$	 $\hat{=}$	$(****)(**)(**)$	-1	$\frac{8!}{4\cdot 2^2\cdot 2!} = 1260$

+ 3p

+ 3p

H4.2

How to get order before constructing table?

(1) $Q_4 \equiv \langle i\sigma_1, i\sigma_2 \mid \sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k \rangle$

	$i\sigma_1$	$i\sigma_2$	$-\mathbb{1}$	$i\sigma_3$	$-i\sigma_3$	$\mathbb{1}$	$-i\sigma_2$	$-i\sigma_1$
$i\sigma_1$	$-\mathbb{1}$	$-i\sigma_3$	$-i\sigma_1$	$i\sigma_2$	$-i\sigma_2$	$i\sigma_1$	$i\sigma_3$	$\mathbb{1}$
$i\sigma_2$	$i\sigma_3$	$-\mathbb{1}$	$-i\sigma_2$	$-i\sigma_1$	$i\sigma_1$	$i\sigma_2$	$\mathbb{1}$	$-i\sigma_3$
$-\mathbb{1}$	$-i\sigma_1$	$-i\sigma_2$	$\mathbb{1}$	$-i\sigma_3$	$i\sigma_3$	$-\mathbb{1}$	$i\sigma_2$	$i\sigma_1$
$i\sigma_3$	$-i\sigma_2$	$i\sigma_1$	$-i\sigma_3$	$-\mathbb{1}$	$\mathbb{1}$	$i\sigma_3$	$-i\sigma_1$	$i\sigma_2$
$-i\sigma_3$	$i\sigma_2$	$-i\sigma_1$	$i\sigma_3$	$\mathbb{1}$	$-\mathbb{1}$	$-i\sigma_3$	$i\sigma_1$	$-i\sigma_2$
$\mathbb{1}$	$i\sigma_1$	$i\sigma_2$	$-\mathbb{1}$	$i\sigma_3$	$-i\sigma_3$	$\mathbb{1}$	$-i\sigma_2$	$-i\sigma_1$
$-i\sigma_2$	$-i\sigma_3$	$\mathbb{1}$	$i\sigma_2$	$i\sigma_1$	$-i\sigma_1$	$-i\sigma_2$	$-\mathbb{1}$	$i\sigma_3$
$-i\sigma_1$	$\mathbb{1}$	$i\sigma_3$	$i\sigma_1$	$-i\sigma_2$	$i\sigma_2$	$-i\sigma_1$	$-i\sigma_3$	$-\mathbb{1}$

- $(i\sigma_1)^2 = -\mathbb{1}$
- $i\sigma_1 i\sigma_2 = -i\sigma_3$
- $i\sigma_2 i\sigma_1$ using $\{\sigma_i, \sigma_j\} = 0$ from $\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$
- $i\sigma_1 i\sigma_3 = -(-i\sigma_2) = i\sigma_2$
- $i\sigma_2 i\sigma_3 = -i\sigma_1$

$|Q_4| = 8$

+ 3p

$\mathrm{ord}(\mathbb{1}) = 1$, $\mathrm{ord}(-\mathbb{1}) = 2$

$\mathrm{Ord}(X) = 4$, $X = \{i\sigma_1, i\sigma_2, i\sigma_3, -i\sigma_3, -i\sigma_2, -i\sigma_1\}$

(2) Conjugacy classes: $gHg^{-1} = H$ or $ghg^{-1} \in H$ $\forall g \in Q_4$, $h \in H$

$[\mathbb{1}] = \{\mathbb{1}\}$ and $[-\mathbb{1}] = \{-\mathbb{1}\}$ trivial

General case for the remaining elements: $\underbrace{(\pm i\sigma_j)}_{g}\underbrace{(\pm i\sigma_i)}_{h}\underbrace{(\mp i\sigma_j)}_{g^{-1}}$

as $g \in \{\mathbb{1}, -\mathbb{1}\}$ trivial as well. $\Rightarrow$ $\sigma_j(\pm i\sigma_i)\sigma_j = \sigma_j^2(\mp i\sigma_i) = \mp i\sigma_i$

+ 3p

$\Rightarrow [i\sigma_1] = \{i\sigma_1, -i\sigma_1\}$, $[i\sigma_2] = \{i\sigma_2, -i\sigma_2\}$, $[i\sigma_3] = \{i\sigma_3, -i\sigma_3\}$

(3) Following from Lagrange's theorem, all SG have to be of orders 1, 2, 4, 8

$H_1 = \{\mathbb{1}\}$, $H_8 = Q_4$; $H_2^{(1)} = \{\mathbb{1}, -\mathbb{1}\}$ as it has to be generated by an element of order 2. All other SG can only be generated by an element of order 4 (see last page) and thus $\cong C_4$:

How to see if we have all SG?

$H_4^{(1)} = \{\mathbb{1}, i\sigma_1, -\mathbb{1}, -i\sigma_1\}$, $H_4^{(2)} = \{\mathbb{1}, i\sigma_2, -\mathbb{1}, -i\sigma_2\}$, $H_4^{(3)} = \{\mathbb{1}, i\sigma_3, -\mathbb{1}, -i\sigma_3\}$

<u>Proper</u> invariant SG ($\neq H_1, H_8$) have to be the union of conjugacy classes:

$H_2^{(1)} = [\mathbb{1}] \cup [-\mathbb{1}]$, $H_4^{(1)} = [\mathbb{1}] \cup [-\mathbb{1}] \cup [i\sigma_1]$,

$H_4^{(2)} = [\mathbb{1}] \cup [-\mathbb{1}] \cup [i\sigma_2]$, $H_4^{(3)} = [\mathbb{1}] \cup [-\mathbb{1}] \cup [i\sigma_3]$ + 3p

(4) $Q_4/H = \{gH \mid g \in Q_4\}$

$Q_4/H_1 = \{\{i\sigma_1\}, \{i\sigma_2\}, \{-\mathbb{1}\}, \{i\sigma_3\}, \{-i\sigma_3\}, \{\mathbb{1}\}, \{-i\sigma_2\}, \{-i\sigma_1\}\}$

$Q_4/H_8 = Q_4$ Identify groups Z_2, Z_4, V ?

Quotient groups also for non-proper?

$Q_4/H_2^{(1)} = \{\{i\sigma_1, -i\sigma_1\}, \{i\sigma_2, -i\sigma_2\}, \{i\sigma_3, -i\sigma_3\}, \{\mathbb{1}, -\mathbb{1}\}\}$

$Q_4/H_4^{(1)} = \{\{\mathbb{1}, i\sigma_1, -\mathbb{1}, -i\sigma_1\}, \{i\sigma_2, i\sigma_3, -i\sigma_2, -i\sigma_3\}\}$

$Q_4/H_4^{(2)} = \{\{\mathbb{1}, i\sigma_2, -\mathbb{1}, -i\sigma_2\}, \{i\sigma_1, -i\sigma_3, -i\sigma_1, i\sigma_3\}\}$

$Q_4/H_4^{(3)} = \{\{\mathbb{1}, i\sigma_3, -\mathbb{1}, -i\sigma_3\}, \{i\sigma_1, i\sigma_2, -i\sigma_1, -i\sigma_2\}\}$ 1p

(5) $D_N = \langle c, b \mid c^N = e, b^2 = e, (bc)^2 = e \rangle$

$(bc)^2 = e = bcbc \Leftrightarrow bcb = c^{-1} \Leftrightarrow bcb^{-1} = c^{-1}$ 1p

(6) D_4 has a generator of order 4 and a generator of order 2.

So $-\mathbb{1}$ is the only choice for the generator of order 2.

Then (5) has to be fulfilled for the other generator for an isomorphism.

Why does this have to hold if we want an isomorphism?

$(-\mathbb{1})c(-\mathbb{1}) \overset{!}{=} c^{-1} \Rightarrow c = c^{-1} \Leftrightarrow c^2 = e$

But we don't have another element of order 2. $\rightsquigarrow$ not possible

$Q_4 = \langle a = i\sigma_1, b = i\sigma_2 \mid a^4 = \mathbb{1}, b^2 = -\mathbb{1}, bab^{-1} = a^{-1} = -a \rangle$

as $(i\sigma_1)^2 = -\mathbb{1}$, $(i\sigma_2)^2 = -\mathbb{1}$

$(i\sigma_2)(i\sigma_1)(i\sigma_2)^{-1} = i\sigma_2\sigma_1\sigma_2 = -i\sigma_1$ + 2p

15 → 95% M.M.