

Disclaimer

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<https://www.physics-and-stuff.com/>

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H5.1

Have an arbitrary basis $\{v_1, \dots, v_n\}$ of an n -dim. vector space V with scalar product $\langle \cdot, \cdot \rangle$, $|v| = \sqrt{\langle v, v \rangle}$

We then construct an orthonormal basis $\{e_1, \dots, e_n\}$

as follows:
$$e_1 := \frac{v_1}{\sqrt{\langle v_1, v_1 \rangle}} \mapsto \langle e_1, e_1 \rangle = \frac{\langle v_1, v_1 \rangle}{\langle v_1, v_1 \rangle} = 1$$

Furthermore, we define

$$e_2 := \frac{v_2 - e_1 \langle e_1, v_2 \rangle}{|v_2 - e_1 \langle e_1, v_2 \rangle|} \mapsto \langle e_1, e_2 \rangle = \frac{\langle e_1, v_2 \rangle - \langle e_1, v_2 \rangle}{|e_1| |v_2 - e_1 \langle e_1, v_2 \rangle|}$$

$$\langle e_2, e_2 \rangle = \frac{\langle v_2 - e_1 \langle e_1, v_2 \rangle, v_2 - e_1 \langle e_1, v_2 \rangle \rangle}{|v_2 - e_1 \langle e_1, v_2 \rangle|^2} = \frac{= 0}{|v_2 - e_1 \langle e_1, v_2 \rangle|^2} = 1$$

We thus prove all e_j , $j \geq 3$ by induction. Assume that we already have a set of orthonormal basis vectors

$\{e_1, \dots, e_n\}$ with $e_i \cdot e_j = \delta_{ij} \forall i, j = 1, \dots, n$. We claim that

$$e_{n+1} := \frac{v_{n+1} - \sum_i e_i \langle e_i, v_{n+1} \rangle}{|v_{n+1} - \sum_i e_i \langle e_i, v_{n+1} \rangle|} \text{ is a choice, which}$$

fulfills $\langle e_{n+1}, e_i \rangle = \delta_{i, n+1} \forall i = 1, \dots, n+1$.

$$\begin{aligned} \mapsto \langle e_{n+1}, e_{n+1} \rangle &= \frac{\langle v_{n+1} - \sum_i e_i \langle e_i, v_{n+1} \rangle, v_{n+1} - \sum_i e_i \langle e_i, v_{n+1} \rangle \rangle}{|v_{n+1} - \sum_i e_i \langle e_i, v_{n+1} \rangle|^2} \\ &= \frac{|v_{n+1} - \sum_i e_i \langle e_i, v_{n+1} \rangle|^2}{|v_{n+1} - \sum_i e_i \langle e_i, v_{n+1} \rangle|^2} = 1 \end{aligned}$$

$$\langle e_{n+1}, e_j \rangle = \frac{\langle v_{n+1}, e_j \rangle - \langle e_j, v_{n+1} \rangle}{|e_j| |v_{n+1} - \sum_i e_i \langle e_i, v_{n+1} \rangle|} = 0$$

□

+

10p

#5.2

$f_1(x,y) = x^2, f_2(x,y) = xy, f_3(x,y) = y^2$

Invariant under all (OC2) trafo's?

a) $\begin{pmatrix} x \\ y \end{pmatrix} \xrightarrow{R} \begin{pmatrix} x' \\ y' \end{pmatrix} = R \begin{pmatrix} x \\ y \end{pmatrix} \rightsquigarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$

$\rightsquigarrow R^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$R^{-1}x, R^{-1}y$ not possible!

$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \xrightarrow{DCR} \begin{pmatrix} p_1(R^{-1}x) \\ p_2(R^{-1}x) \\ p_3(R^{-1}x) \end{pmatrix} = \begin{pmatrix} p_1 \\ -p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$

$\rightsquigarrow DCR = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ + 5p

b) $\begin{pmatrix} x' \\ y' \end{pmatrix} = R \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1/2x - \sqrt{3}/2y \\ \sqrt{3}/2x - 1/2y \end{pmatrix}$

Calculate R^{-1} :

$\begin{array}{cc|cc} -1/2 & -\sqrt{3}/2 & 1 & 0 \\ \sqrt{3}/2 & -1/2 & 0 & 1 \end{array} \rightsquigarrow \begin{array}{cc|cc} -1/2 & -\sqrt{3}/2 & 1 & 0 \\ 0 & -2 & \sqrt{3} & 1 \end{array}$

$\rightsquigarrow \begin{array}{cc|cc} -1/2 & 0 & 1/4 & -\sqrt{3}/4 \\ 0 & -2 & \sqrt{3} & 1 \end{array} \rightsquigarrow \begin{array}{cc|cc} 1 & 0 & -1/2 & \sqrt{3}/2 \\ 0 & 1 & -\sqrt{3}/2 & -1/2 \end{array}$

$\rightsquigarrow R^{-1} = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}$

$R^{-1}x$ or $R^{-1}x'$?

$\rightsquigarrow \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \xrightarrow{DCR} \begin{pmatrix} p_1(R^{-1}x) \\ p_2(R^{-1}x) \\ p_3(R^{-1}x) \end{pmatrix}, R^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1/2x + \sqrt{3}/2y \\ -\sqrt{3}/2x - 1/2y \end{pmatrix}$

$= \begin{pmatrix} (-1/2x + \sqrt{3}/2y)^2 \\ (-1/2x + \sqrt{3}/2y)(-\sqrt{3}/2x - 1/2y) \\ (-\sqrt{3}/2x - 1/2y)^2 \end{pmatrix}$

$= \begin{pmatrix} 1/4x^2 + 3/4y^2 - \sqrt{3}/4xy \\ \sqrt{3}/4x^2 + \sqrt{3}/4y^2 - 1/2xy \\ 3/4x^2 + 1/4y^2 + \sqrt{3}/4xy \end{pmatrix} = \begin{pmatrix} 1/4 & -\sqrt{3}/4 & 3/4 \\ \sqrt{3}/4 & -1/2 & \sqrt{3}/4 \\ 3/4 & \sqrt{3}/4 & 1/4 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$

+ 2p

$$\mapsto DCR^T = \begin{pmatrix} 1/4 & -\sqrt{3}/4 & 3/4 \\ \sqrt{3}/4 & -1/2 & \sqrt{3}/4 \\ 3/4 & \sqrt{3}/4 & 1/4 \end{pmatrix}$$

That is basis transformation

$\Phi(R)$ for coordinates

← looks like \otimes^T .

$17p \rightarrow 85\%$ k.k.