20.11.2017 Group theory Exercise 5 Marvin Zanke

H5.1

Have an arbitrary basis $\{v_1, \dots, v_n\}$ of an n-dim. vector space V with scalar product $\langle\cdot,\cdot\rangle$, $|v| = \sqrt{\langle v,v\rangle}$

We then construct an orthonormal basis $\{e_1, \dots, e_n\}$ as follows:

$$\boxed{e_1 := \frac{v_1}{\sqrt{\langle v_1, v_1\rangle}}} \Rightarrow \langle e_1, e_1\rangle = \frac{\langle v_1, v_1\rangle}{\langle v_1, v_1\rangle} = 1$$

Furthermore, we define

$$e_2 := \frac{v_2 - e_1\langle e_1, v_2\rangle}{|v_2 - e_1\langle e_1, v_2\rangle|} \Rightarrow \langle e_1, e_2\rangle = \frac{\langle e_1, v_2\rangle - \langle e_1, v_2\rangle}{|e_1||v_2 - e_1\langle e_1, v_2\rangle|} = 0$$

$$\langle e_2, e_2\rangle = \frac{\langle v_2 - e_1\langle e_1, v_2\rangle, v_2 - e_1\langle e_1, v_2\rangle\rangle}{|v_2 - e_1\langle e_1, v_2\rangle|^2} = \frac{|v_2 - e_1\langle e_1, v_2\rangle|^2}{|v_2 - e_1\langle e_1, v_2\rangle|^2} = 1$$

We thus prove all e_j, $j \geq 3$ by induction. Assume that we already have a set of orthonormal basis vectors $\{e_1, \dots, e_n\}$ with $e_i \cdot e_j = \delta_{ij}$ $\forall i,j = 1, \dots, n$. We claim that

$$\boxed{e_{n+1} := \frac{v_{n+1} - \sum_i e_i\langle e_i, v_{n+1}\rangle}{|v_{n+1} - \sum_i e_i\langle e_i, v_{n+1}\rangle|}}$$

is a choice, which fulfills $\langle e_{n+1}, e_i\rangle = \delta_{i\,n+1}$ $\forall i = 1, \dots, n+1$.

$$\Rightarrow \langle e_{n+1}, e_{n+1}\rangle = \frac{\langle v_{n+1} - \sum_i e_i\langle e_i, v_{n+1}\rangle, v_{n+1} - \sum_i e_i\langle e_i, v_{n+1}\rangle\rangle}{|v_{n+1} - \sum_i e_i\langle e_i, v_{n+1}\rangle|^2}$$

$$= \frac{|v_{n+1} - \sum_i e_i\langle e_i, v_{n+1}\rangle|^2}{|v_{n+1} - \sum_i e_i\langle e_i, v_{n+1}\rangle|^2} = 1$$

$$\cdot \langle e_{n+1}, e_j\rangle = \frac{\langle v_{n+1}, e_j\rangle - \langle e_j, v_{n+1}\rangle}{|e_j||v_{n+1} - \sum_i e_i\langle e_i, v_{n+1}\rangle|} = 0$$

MZ

+ 10p

H5.2

$$f_1(x,y) = x^2, \quad f_2(x,y) = xy, \quad f_3(x,y) = y^2$$

Invariant under all O(2) trafos?

a)

$$\begin{pmatrix} x \\ y \end{pmatrix} \xrightarrow{R} \begin{pmatrix} x' \\ y' \end{pmatrix} = R\begin{pmatrix} x \\ y \end{pmatrix} \rightsquigarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$$

$$\rightsquigarrow R^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$R^{-1}x, R^{-1}y$ not possible?

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \xrightarrow{D(R)} \begin{pmatrix} f_1(R^{-1}\vec{x}) \\ f_2(R^{-1}\vec{x}) \\ f_3(R^{-1}\vec{x}) \end{pmatrix} = \begin{pmatrix} f_1 \\ -f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

$$\Longrightarrow D(R) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

+ 5p

b)

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = R\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1/2\, x - \sqrt{3}/2\, y \\ \sqrt{3}/2\, x - 1/2\, y \end{pmatrix}$$

Calculate R^{-1}:

$$\left(\begin{array}{cc|cc} -1/2 & -\sqrt{3}/2 & 1 & 0 \\ \sqrt{3}/2 & -1/2 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|cc} -1/2 & -\sqrt{3}/2 & 1 & 0 \\ 0 & -2 & \sqrt{3} & 1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cc|cc} -1/2 & 0 & 1/4 & -\sqrt{3}/4 \\ 0 & -2 & \sqrt{3} & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|cc} 1 & 0 & -1/2 & \sqrt{3}/2 \\ 0 & 1 & -\sqrt{3}/2 & -1/2 \end{array}\right)$$

$$\rightsquigarrow R^{-1} = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}$$

$R^{-1}\vec{x}$ or $R^{-1}\vec{x}'$?

$$\rightsquigarrow \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \xrightarrow{D(R)} \begin{pmatrix} f_1(R^{-1}\vec{x}) \\ f_2(R^{-1}\vec{x}) \\ f_3(R^{-1}\vec{x}) \end{pmatrix}, \quad R^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1/2\, x + \sqrt{3}/2\, y \\ -\sqrt{3}/2\, x - 1/2\, y \end{pmatrix}$$

$$= \begin{pmatrix} \left(-1/2\, x + \sqrt{3}/2\, y\right)^2 \\ \left(-1/2\, x + \sqrt{3}/2\, y\right)\left(-\sqrt{3}/2\, x - 1/2\, y\right) \\ \left(-\sqrt{3}/2\, x - 1/2\, y\right)^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1/4\, x^2 + 3/4\, y^2 - \sqrt{3}/\cancel{4}\,2\, xy \\ \sqrt{3}/4\, x^2 + \sqrt{3}/4\, y^2 - 1/2\, xy \\ 3/4\, x^2 + 1/4\, y^2 + \sqrt{3}/\cancel{4}\,2\, xy \end{pmatrix} = \begin{pmatrix} 1/4 & -\sqrt{3}/\cancel{4}\,2 & 3/4 \\ \sqrt{3}/4\;4 & -1/2 & \sqrt{3}/4 \\ 3/4 & \sqrt{3}/\cancel{4}\,2 & 1/4 \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

+ 2p

$$\rightsquigarrow D^T(R) = \begin{pmatrix} 1/4 & -\sqrt{3}/2 & 3/4 \\ \sqrt{3}/4 & -1/2 & \sqrt{3}/4 \\ 3/4 & \sqrt{3}/2 & 1/4 \end{pmatrix}$$

That is basis transformation $D(R)$ for coordinates looks like D^T.

17p → 85% M.M.