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I raise no claim to correctness and completeness of the given solutions! This equally applies to the corrections mentioned above.

29.11.2017 **Group theory Exercise 6** Marvin Zanke

H6.1

(1) Representation D^G on L (with dimension d) of a group G is irreducible if it only has the invariant subspaces L and $\{0\}$.
Want to show: irrep of an abelian group $\Rightarrow$ one-dimensional

$$D^G(g_1)D^G(g_2) = D^G(g_1 g_2) = D^G(g_2 g_1) = D^G(g_2)D^G(g_1)$$
$\forall g_1, g_2 \in G$

$$\Rightarrow \quad D^G(g)D^G(g') = D^G(g')D^G(g) \quad \forall g, g' \in G$$

$\boxed{\text{Schurs } \Rightarrow \text{ first Lemma}}$ $D^G(g') = \lambda_{g'} \mathbb{1} \quad \forall g' \in G$

second + because the matrix relate same reps matrices.

If we had a representation with $d > 1$, it would be reducible, as $D^G(g') = \lambda_{g'} \mathbb{1}_d$ consists of d irreducible representations.

4p

(2) Cyclic group $G_n = \langle c \mid c^n = e \rangle$ with order n is an abelian group. Every irreducible repr. is thus one-dimensional (see (1)). As we want our repr. to be unitary, it must hold that $D^G(g)^* = D^G(g)^{-1}$ and thus $|D^G(g)|^2 = 1$. It can thus be written as an exponential function in the complex numbers.

set of all unitary irrep.

$l \neq 1$? trivial mapping?

We want it to fulfill $D^G(e) = D^G(c^n) = (D^G(c))^n$ +
and thus
$$D^G_l(c) = e^{2\pi i \frac{l-1}{n}} \quad \text{with } l = 1, \ldots, n$$
the roots of unity.

It follows that

$$D^G_l(c^m) = (D^G_l(c))^m = e^{2\pi i m \frac{l-1}{n}}, \quad m = 1, \ldots, n$$

+ 4p

Why 6-dim? 6x6 matrices or 6 matrices?

H6.2

6-dimensional space of polynomials of degree 2 in (x,y).

$$f(x,y) = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2$$

$$D(e) = \mathbb{1}, \quad D(b) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad D(c) = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$$

$$D(c^2) = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}$$

$$\rightsquigarrow D(e)^{-1} = \mathbb{1}, \quad D(b)^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad D(c)^{-1} = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}$$

$$D(c^2)^{-1} = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$$

Basis: $f_1(x,y) = 1, \quad f_2(-) = x, \quad f_3(-) = y$

$f_4(-) = x^2, \quad f_5(-) = xy, \quad f_6(-) = y^2$

Why only 1- and 2-dim. repr.?

Representation for $D(b)$: $\quad D(b)^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{pmatrix}^T \xrightarrow{D(b)} \begin{pmatrix} f_1(D(b)^{-1}\vec{x}) \\ f_2(\quad " \quad) \\ f_3(\quad " \quad) \\ f_4(\quad " \quad) \\ f_5(\quad " \quad) \\ f_6(\quad " \quad) \end{pmatrix}^T = \begin{pmatrix} f_1 \\ f_2 \\ -f_3 \\ f_4 \\ -f_5 \\ f_6 \end{pmatrix}^T$$

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{pmatrix}^T \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}}_{D_6(b)}$$

+

Representation for $D(c)$: $\quad D(c)^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1/2\, x + \sqrt{3}/2\, y \\ -\sqrt{3}/2\, x - 1/2\, y \end{pmatrix}$ +

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{pmatrix}^T \xrightarrow{D(c)} \begin{pmatrix} f_1(D(c)^{-1}\vec{x}) \\ f_2(D(c)^{-1}\vec{x}) \\ f_3(\sim) \\ f_4(\quad " \quad) \\ f_5(\quad " \quad) \\ f_6(\quad " \quad) \end{pmatrix}^T = \begin{pmatrix} f_1 \\ -1/2 f_2 + \sqrt{3}/2 f_3 \\ -\sqrt{3}/2 f_2 - 1/2 f_3 \\ 1/4 f_4 - \sqrt{3}/2 f_5 + 3/4 f_6 \\ \sqrt{3}/4 f_4 - 1/2 f_5 - \sqrt{3}/4 f_6 \\ 3/4 f_4 + \sqrt{3}/2 f_5 + 1/4 f_6 \end{pmatrix}^T$$

$$= \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{pmatrix}^T \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1/2 & -\sqrt{3}/2 & 0 & 0 & 0 \\ 0 & \sqrt{3}/2 & -1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/4 & \sqrt{3}/4 & 3/4 \\ 0 & 0 & 0 & -\sqrt{3}/2 & -1/2 & \sqrt{3}/2 \\ 0 & 0 & 0 & 3/4 & -\sqrt{3}/4 & 1/4 \end{pmatrix}}_{D_6(c)}$$

Represantation for $D_6(c^2)$: $D(c^2)^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1/2\, x - \sqrt{3}/2\, y \\ \sqrt{3}/2\, x - 1/2\, y \end{pmatrix}$

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{pmatrix}^T \xmapsto{D(c^2)} \begin{pmatrix} f_1(D(c^2)^{-1}\vec{x}) \\ f_2(\quad " \quad) \\ f_3(\quad " \quad) \\ f_4(\quad " \quad) \\ f_5(\quad " \quad) \\ f_6(\quad " \quad) \end{pmatrix}^T = \begin{pmatrix} f_1 \\ -1/2 f_2 - \sqrt{3}/2 f_3 \\ \sqrt{3}/2 f_2 - 1/2 f_3 \\ 1/4 f_4 + \sqrt{3}/2 f_5 + 3/4 f_6 \\ -\sqrt{3}/4 f_4 - 1/2 f_5 + \sqrt{3}/4 f_6 \\ 3/4 f_4 - \sqrt{3}/2 f_5 + 1/4 f_6 \end{pmatrix}^T$$

$$= \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{pmatrix}^T \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1/2 & \sqrt{3}/2 & 0 & 0 & 0 \\ 0 & -\sqrt{3}/2 & -1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/4 & -\sqrt{3}/4 & 3/4 \\ 0 & 0 & 0 & \sqrt{3}/2 & -1/2 & -\sqrt{3}/2 \\ 0 & 0 & 0 & 3/4 & \sqrt{3}/4 & 1/4 \end{pmatrix}}_{D_6(c^2)}$$

$$\rightsquigarrow \quad D_6(bc) = D_6(b)\, D_6(c) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1/2 & -\sqrt{3}/2 & 0 & 0 & 0 \\ 0 & -\sqrt{3}/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/4 & \sqrt{3}/4 & 3/4 \\ 0 & 0 & 0 & \sqrt{3}/2 & 1/2 & -\sqrt{3}/2 \\ 0 & 0 & 0 & 3/4 & \sqrt{3}/4 & 1/4 \end{pmatrix}$$

$$D_6(bc^2) = D_6(b)\, D_6(c^2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1/2 & -\sqrt{3}/2 & 0 & 0 & 0 \\ 0 & \sqrt{3}/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/4 & \sqrt{3}/4 & 3/4 \\ 0 & 0 & 0 & -\sqrt{3}/2 & 1/2 & +\sqrt{3}/2 \\ 0 & 0 & 0 & 3/4 & \sqrt{3}/4 & 1/4 \end{pmatrix}$$

We can now decompose the reducible representation as follows:

$$D_6(e) = \begin{pmatrix} D_6^{(1)}(e) & 0 & 0 & 0 & 0 & 0 \\ 0 & \multicolumn{2}{c}{D_6^{(2)}(e)} & 0 & 0 & 0 \\ 0 & & & 0 & 0 & 0 \\ 0 & 0 & 0 & & & \\ 0 & 0 & 0 & \multicolumn{3}{c}{D_6^{(3)}(e)} \\ 0 & 0 & 0 & & & \end{pmatrix} \qquad D_6^{(1)}(e) = (1)$$

$$D_6^{(2)}(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

+

$$D_6^{(3)}(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$D_6(b) = \begin{pmatrix} D_6^{(1)}(b) & 0 & 0 & 0 & 0 & 0 \\ 0 & \multicolumn{2}{c}{D_6^{(2)}(b)} & 0 & 0 & 0 \\ 0 & & & 0 & 0 & 0 \\ 0 & 0 & 0 & & & \\ 0 & 0 & 0 & \multicolumn{3}{c}{D_6^{(3)}(b)} \\ 0 & 0 & 0 & & & \end{pmatrix} \qquad D_6^{(1)}(b) = (1)$$

$$D_6^{(2)}(b) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

+

$$D_6^{(3)}(b) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$D_6(c) = \begin{pmatrix} D_6^{(1)}(c) & 0 & 0 & 0 & 0 & 0 \\ 0 & \multicolumn{2}{c}{D_6^{(2)}(c)} & 0 & 0 & 0 \\ 0 & & & 0 & 0 & 0 \\ 0 & 0 & 0 & & & \\ 0 & 0 & 0 & \multicolumn{3}{c}{D_6^{(3)}(c)} \\ 0 & 0 & 0 & & & \end{pmatrix} \qquad D_6^{(1)}(c) = (1)$$

$$D_6^{(2)}(c) = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$$

+

$$D_6^{(3)}(c) = \begin{pmatrix} 1/4 & \sqrt{3}/4 & 3/4 \\ -\frac{\sqrt{3}}{2} & -1/2 & \sqrt{3}/2 \\ 3/4 & -\sqrt{3}/4 & 1/4 \end{pmatrix}$$

Why can we use the $1/N_G \sum |\chi(g)|^2 = 1$ theorem already for the subspaces $D^{(1)}, D^{(2)}$ etc.? i.e. Why can we split into $\begin{pmatrix} D^1(g) & C \\ 0 & D^2(g) \end{pmatrix}$ and then consider each for itself?

$$D_6(c^2) = \begin{pmatrix} D_6^{(1)}(c^2) & 0 & 0 & 0 & 0 & 0 \\ 0 & \multicolumn{2}{c}{D_6^{(2)}(c^2)} & 0 & 0 & 0 \\ 0 & & & 0 & 0 & 0 \\ 0 & 0 & 0 & & & \\ 0 & 0 & 0 & \multicolumn{3}{c}{D_6^{(3)}(c^2)} \\ 0 & 0 & 0 & & & \end{pmatrix} \qquad D_6^{(1)}(c^2) = (1)$$

$$D_6^{(2)}(c^2) = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}$$

+

$$D_6^{(3)}(c^2) = \begin{pmatrix} 1/4 & -\sqrt{3}/4 & 3/4 \\ \sqrt{3}/2 & -1/2 & -\frac{\sqrt{3}}{2} \\ 3/4 & \sqrt{3}/4 & 1/4 \end{pmatrix}$$

$$D_6(bc) = \begin{pmatrix} D_6^{(1)}(bc) & 0 & 0 & 0 & 0 & 0 \\ 0 & \multicolumn{2}{c}{D_6^{(2)}(bc)} & 0 & 0 & 0 \\ 0 & & & 0 & 0 & 0 \\ 0 & 0 & 0 & & & \\ 0 & 0 & 0 & & D_6^{(3)}(bc) & \\ 0 & 0 & 0 & & & \end{pmatrix}$$

$$D_6^{(1)}(bc) = (1)$$

$$D_6^{(2)}(bc) = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}$$

$$D_6^{(3)}(bc) = \begin{pmatrix} 1/4 & \sqrt{3}/4 & 3/4 \\ \sqrt{3}/2 & 1/2 & -\sqrt{3}/2 \\ 3/4 & \sqrt{3}/4 & 1/4 \end{pmatrix}$$

$$D_6(bc^2) = \begin{pmatrix} D_6^{(1)}(bc^2) & 0 & 0 & 0 & 0 & 0 \\ 0 & \multicolumn{2}{c}{D_6^{(2)}(bc^2)} & 0 & 0 & 0 \\ 0 & & & 0 & 0 & 0 \\ 0 & 0 & 0 & & & \\ 0 & 0 & 0 & & D_6^{(3)}(bc^2) & \\ 0 & 0 & 0 & & & \end{pmatrix}$$

$$D_6^{(1)}(bc^2) = (1)$$

$$D_6^{(2)}(bc^2) = \begin{pmatrix} -1/2 & +\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$$

$$D_6^{(3)}(bc^2) = \cancel{\begin{pmatrix} 1/4 & \sqrt{3}/4 & 3/4 \\ -\sqrt{3}/2 & 1/2 & \sqrt{3}/2 \\ 3/4 & \sqrt{3}/4 & 1/4 \end{pmatrix}}$$

Check by matrix multiplication.

As the dimension of the representation $D_6^{(1)}(g)$ is equal to 1, it has to be an irreducible representation and thus $L_1 = \left\langle \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix} \right\rangle$ is an invariant subspace.

$$\begin{pmatrix} 1/4 & -\sqrt{3}/4 & 3/4 \\ -\sqrt{3}/2 & 1/2 & \sqrt{3}/2 \\ 3/4 & \sqrt{3}/4 & 1/4 \end{pmatrix}$$

To check wether the other representations are irreducible, we make use of the theorem that $\frac{1}{N_G}\sum_{g\in G} |\chi(g)|^2 \overset{!}{=} 1$ for irreducible representations, where $\chi(g) := \mathrm{Tr}(D(g))$

$$D_6^{(2)}: \frac{1}{6}\left\{(\mathrm{Tr}(D_6^{(2)}(e)))^2 + \dots\right\} = \frac{1}{6}\left\{2^2 + 0^2 + (-1)^2 + (-1)^2 + 0^2 + 0^2\right\}$$
$$= 1.$$

This can be seen quicker, if one recalls that conjugacy classes (i.e. elements from the same conjugacy class) have the same character and $[e] = \{e\}$, $[c] = \{c, c^2\}$, $[b] = \{b, bc, bc^2\}$

Now consider

$$D_6^{(3)}: \frac{1}{6}\left\{9 + 3\cdot(1)^2 + 2\cdot(0)^2\right\} = 2$$

Thus, $D_6^{(3)}(g)$ is further reducible !

To find the irreducible representations, we have to construct a new basis. We claim that for the subspace on which $D_6^{(3)}(g)$ acts, the following vectors do the job:

Formal way to derive this new basis?

$$\tilde{f}_4 := x^2+y^2 \;,\; \tilde{f}_5 := x^2-y^2 \;,\; \tilde{f}_6 := xy \qquad + \; 3p$$

It's obviously an equivalent basis, as $f_4 = \frac{1}{2}(\tilde{f}_4+\tilde{f}_5)$ and $f_5 = \frac{1}{2}(\tilde{f}_4 - \tilde{f}_5)$

What if wrote down f_5 as $\tilde{f}_4$ and vice versa? → no diagonal form ?!

Then $$\begin{pmatrix}\tilde{f}_4\\ \tilde{f}_5\\ \tilde{f}_6\end{pmatrix}^T \xmapsto{D(b)} \begin{pmatrix}\tilde{f}_4(R^{-1}x)\\ \tilde{f}_5(R^{-1}x)\\ \tilde{f}_6(R^{-1}x)\end{pmatrix}^T = \begin{pmatrix}\tilde{f}_4\\ \tilde{f}_5\\ -\tilde{f}_6\end{pmatrix}^T = \begin{pmatrix}\tilde{f}_4\\ \tilde{f}_5\\ \tilde{f}_6\end{pmatrix}^T \underbrace{\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}}_{\tilde{D}_6^{(3)}(b)}$$

$$\begin{pmatrix}\tilde{f}_4\\ \tilde{f}_5\\ \tilde{f}_6\end{pmatrix}^T \xmapsto{D(c)} \begin{pmatrix}\tilde{f}_4(R^{-1}x)\\ \tilde{f}_5(R^{-1}x)\\ \tilde{f}_6(R^{-1}x)\end{pmatrix}^T = \begin{pmatrix}x^2+y^2\\ -\frac{1}{2}x^2+\frac{1}{2}y^2-\sqrt{3}xy\\ \frac{\sqrt{3}}{4}x^2-\frac{\sqrt{3}}{4}y^2-\frac{1}{2}xy\end{pmatrix}^T = \begin{pmatrix}\tilde{f}_4\\ -\frac{1}{2}\tilde{f}_5-\sqrt{3}\tilde{f}_6\\ \frac{\sqrt{3}}{4}\tilde{f}_5-\frac{1}{2}\tilde{f}_6\end{pmatrix}^T$$

$$= \begin{pmatrix}\tilde{f}_4\\ \tilde{f}_5\\ \tilde{f}_6\end{pmatrix}^T \underbrace{\begin{pmatrix}1&0&0\\0&-\frac{1}{2}&\frac{\sqrt{3}}{4}\\0&-\sqrt{3}&-\frac{1}{2}\end{pmatrix}}_{\tilde{D}_6^{(3)}(c)}$$

$$\begin{pmatrix}\tilde{f}_4\\ \tilde{f}_5\\ \tilde{f}_6\end{pmatrix}^T \xmapsto{D(c^2)} \begin{pmatrix}\tilde{f}_4(R^{-1}x)\\ \tilde{f}_5(R^{-1}x)\\ \tilde{f}_6(R^{-1}x)\end{pmatrix}^T = \begin{pmatrix}x^2+y^2\\ -\frac{1}{2}x^2+\frac{1}{2}y^2+\sqrt{3}xy\\ -\frac{\sqrt{3}}{4}x^2+\frac{\sqrt{3}}{4}y^2-\frac{1}{2}xy\end{pmatrix}^T = \begin{pmatrix}\tilde{f}_4\\ -\frac{1}{2}\tilde{f}_5+\sqrt{3}\tilde{f}_6\\ -\frac{\sqrt{3}}{4}\tilde{f}_5-\frac{1}{2}\tilde{f}_6\end{pmatrix}^T$$

$$= \begin{pmatrix}\tilde{f}_4\\ \tilde{f}_5\\ \tilde{f}_6\end{pmatrix}^T \underbrace{\begin{pmatrix}1&0&0\\0&-\frac{1}{2}&-\frac{\sqrt{3}}{4}\\0&\sqrt{3}&-\frac{1}{2}\end{pmatrix}}_{\tilde{D}_6^{(3)}(c^2)}$$

$$\rightsquigarrow \tilde{D}_6^{(3)}(bc) = \begin{pmatrix}1&0&0\\0&-\frac{1}{2}&\frac{\sqrt{3}}{4}\\0&\sqrt{3}&\frac{1}{2}\end{pmatrix} \;,\; \tilde{D}_6^{(3)}(bc^2) = \begin{pmatrix}1&0&0\\0&-\frac{1}{2}&-\frac{\sqrt{3}}{4}\\0&-\sqrt{3}&\frac{1}{2}\end{pmatrix}$$

$$\rightsquigarrow \quad \tilde{D}_6^{(3)}(e) = \begin{pmatrix} d_6^{(1)}(e) & 0 & 0 \\ 0 & & \\ 0 & & d_6^{(2)}(e) \end{pmatrix}, \quad d_6^{(1)}(e) = (1), \quad d_6^{(2)}(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\tilde{D}_6^{(3)}(b) = \begin{pmatrix} d_6^{(1)}(b) & 0 & 0 \\ 0 & & \\ 0 & & d_6^{(2)}(b) \end{pmatrix}, \quad d_6^{(1)}(b) = (1), \quad d_6^{(2)}(b) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\tilde{D}_6^{(3)}(c) = \begin{pmatrix} d_6^{(1)}(c) & 0 & 0 \\ 0 & & \\ 0 & & d_6^{(2)}(c) \end{pmatrix}, \quad d_6^{(1)}(c) = (1), \quad d_6^{(2)}(c) = \begin{pmatrix} -1/2 & \sqrt{3}/4 \\ -\sqrt{3} & -1/2 \end{pmatrix}$$

$$\tilde{D}_6^{(3)}(c^2) = \begin{pmatrix} d_6^{(1)}(c^2) & 0 & 0 \\ 0 & & \\ 0 & & d_6^{(2)}(c^2) \end{pmatrix}, \quad d_6^{(1)}(c^2) = (1), \quad d_6^{(2)}(c^2) = \begin{pmatrix} -1/2 & -\sqrt{3}/4 \\ \sqrt{3} & -1/2 \end{pmatrix}$$

$$\tilde{D}_6^{(3)}(bc) = \begin{pmatrix} d_6^{(1)}(bc) & 0 & 0 \\ 0 & & \\ 0 & & d_6^{(2)}(bc) \end{pmatrix}, \quad d_6^{(1)}(bc) = (1), \quad d_6^{(2)}(bc) = \begin{pmatrix} -1/2 & \sqrt{3}/4 \\ \sqrt{3} & 1/2 \end{pmatrix}$$

$$\tilde{D}_6^{(3)}(bc^2) = \begin{pmatrix} d_6^{(1)}(bc^2) & 0 & 0 \\ 0 & & \\ 0 & & d_6^{(2)}(bc^2) \end{pmatrix}, \quad d_6^{(1)}(bc^2) = (1), \quad d_6^{(2)}(bc^2) = \begin{pmatrix} -1/2 & -\sqrt{3}/4 \\ -\sqrt{3} & 1/2 \end{pmatrix}$$

All in all, we have the invariant subspaces:

$S_1 = \langle f_1 \rangle = \langle 1 \rangle$, $S_2 = \langle f_2, f_3 \rangle = \langle x, y \rangle$

$S_3 = \langle \tilde{f}_4 \rangle = \langle x^2+y^2 \rangle$, $S_4 = \langle \tilde{f}_5, \tilde{f}_6 \rangle = \langle x^2-y^2, xy \rangle$

Did I really choose the correct way of multiplying the basis (from the left) now? And also considered the $R^{-1}x$ stuff correctly? Is there a formal way to find the new basis $\tilde{f}_4 = x^2+y^2$, $\tilde{f}_5 = \_\_\_\_$?

– yes
– orthogonalization of matrix?
but it is not easier!

19p → 95%. M.M.