Disclaimer

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https://www.physics-and-stuff.com/

I raise no claim to correctness and completeness of the given solutions! This equally applies to the corrections mentioned above.

04.12.2017 Group theory Exercise 7 Marvin Zanke

H7.1

(1) $j_\pm^{(1)} = \mp \frac{1}{\sqrt{2}} j_\pm \quad , \quad j_\pm = j_x \pm i j_y$

$j_0^{(1)} = j_z \quad , \quad [j_z, j_\pm] = \pm j_\pm \quad , \quad [j_+, j_-] = 2 j_z$

Consider the rotation group in order to show that this is an irreducible spherical tensor of rank 1, $S_\varepsilon = 1\!\!1 - \frac{i}{\hbar} \epsilon \hat{n} \cdot \vec{J}$

From the lecture, we take the following conditions, which then have to be fulfilled,

where from rel. $[j_z, j_\pm] = \ldots$ etc.?

- $[j_z, j_m^{(1)}] = \hbar m j_m^{(1)}$
- $[j_\pm, j_m^{(1)}] = \hbar \sqrt{2 - m(m \pm 1)}\, j_{m\pm1}^{(1)}$

} set $\hbar = 1$ from now on

$\hbar = 1$ in this exercise? Otherwise we have often missing factors of $\hbar$?

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Check those,

$[j_z, j_0^{(1)}] = [j_z, j_z] = 0 = 0 \cdot j_0^{(1)}$ ✓

$[j_z, j_{\pm1}^{(1)}] = [j_z, \mp \frac{1}{\sqrt{2}} j_\pm] = \mp \frac{1}{\sqrt{2}} [j_z, j_\pm]$

$= \mp \frac{1}{\sqrt{2}} (\pm j_\pm) = \pm \frac{1}{\sqrt{2}} (\mp j_\pm) = \pm j_{\pm1}^{(1)} = m j_\pm^{(1)}$ ✓

$[j_\pm, j_0^{(1)}] = [j_\pm, j_z] = -(\pm j_\pm) = \mp j_\pm = \sqrt{2} \left(\mp \frac{1}{\sqrt{2}} j_\pm\right)$

$= \sqrt{2}\, j_{\pm1}^{(1)} = \sqrt{2 - 0(\pm1)}\, j_{m\pm1}^{(1)}$ ✓

$[j_\pm, j_{+1}^{(1)}] = [j_\pm, -\frac{1}{\sqrt{2}} j_+] = -\frac{1}{\sqrt{2}} [j_\pm, j_+] = \begin{cases} 0 & \text{for upper sign} \\ \underbrace{-\frac{1}{\sqrt{2}}(-2 j_z)}_{\sqrt{2}\, j_0^{(1)}}, & \text{lower} \end{cases}$

$= \begin{cases} \sqrt{2 - 1\cdot(2)}\, j_2^{(1)} & \text{upper sign} \\ \sqrt{2 - 1\cdot(0)}\, j_0^{(1)} & \text{lower sign} \end{cases}$ ✓

$[j_\pm, j_{-1}^{(1)}] = [j_\pm, \frac{1}{\sqrt{2}} j_-] = \frac{1}{\sqrt{2}} [j_\pm, j_-] = \begin{cases} \sqrt{2}\, j_z & \text{, upper sign} \\ 0 & \text{, lower sign} \end{cases}$

$= \begin{cases} \sqrt{2 - (-1)(0)}\, j_0^{(1)} & \text{, upper sign} \\ \sqrt{2 - (-1)(-2)}\, j_{-2}^{(1)} & \text{, lower sign} \end{cases}$ ✓

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(2) Wigner-Eckart theorem states:

$$\langle \xi'; j' m' | j_q^{(1)} | \xi; j m \rangle = (-1)^{2j'-2j} \frac{\langle j 1; m q | j 1; j' m' \rangle}{\sqrt{(2j+1)}} \langle \xi'; j' \| j^1 \| \xi; j \rangle$$

$q \in \{-1, 0, 1\}$

Where from coefficient? In Wigner-Eckart $\langle \beta j m | \alpha_j \beta_l \rangle$?

Consider $q = 0$: $(j_0^{(1)} = j_z)$

$$\rightsquigarrow \langle \xi'; j' m' | \hbar m | \xi; j m \rangle = (-1)^{2j'-2j} \frac{\langle j 1; m 0 | j 1; j' m' \rangle}{\sqrt{(2j+1)}} \langle \xi'; j' \| j^1 \| \xi; j \rangle$$

$$\Leftrightarrow \langle \xi'; j' \| j^1 \| \xi; j \rangle = (-1)^{2j'-2j} \sqrt{(2j+1)}\, \hbar m \frac{\langle \xi'; j' m' | \xi; j m \rangle}{\langle j 1; m 0 | j 1; j' m' \rangle} \leftarrow \begin{matrix} \xi' = \xi \\ j' = j \\ m' = m \end{matrix}$$

How to get the δ-distr. to the other side; i.e. $\xi' = \xi$ etc. on RHS as well?

$$\Leftrightarrow \langle \xi'; j' \| j^1 \| \xi, j \rangle = \sqrt{(2j+1)}\, \hbar m \frac{1}{\langle j 1; m 0 | j 1; j m \rangle}$$

$$\overset{m=j}{\rightsquigarrow} \langle \xi'; j' \| j^1 \| \xi, j \rangle = \hbar j \sqrt{(2j+1)} \frac{1}{\sqrt{\frac{j}{j+1}}}$$

$$= \hbar \sqrt{j (j+1) (2j+1)}$$

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5P

LHS still depends on ξ' and j'?

$q = \pm 1$ not needed?

H7.2

In A7.1, we derived

Also consider $j=0$?

$$d^{(j)}_{m'm} = \sum_{\substack{m_1 m_2 \\ m_1' m_2'}} \langle jm' | j_1 m_1' ; j_2 m_2' \rangle \langle j_1 m_1 ; j_2 m_2 | jm \rangle d^{(j_1)}_{m_1' m_1} d^{(j_2)}_{m_2' m_2}$$

$$d^{(1)}_{11} = \frac{1}{2}(1+\cos\beta) \text{ from A7.1}$$

For the remaining elements, ($d^{(1/2)}_{\pm 1/2 \pm 1/2} = \cos\beta/2$, $d^{(1/2)}_{\pm 1/2 \mp 1/2} = \mp \sin\beta/2$)

$$d^{(1)}_{+1-1} = \sum_{\substack{m_1 m_2 \\ m_1' m_2'}} \langle 1\,1 | \tfrac12 m_1' ; \tfrac12 m_2' \rangle \langle \tfrac12 m_1 ; \tfrac12 m_2 | 1\, -1 \rangle d^{(1/2)}_{m_1' m_1} d^{1/2}_{m_2' m_2}$$

$$= d^{(1/2)}_{1/2 -1/2} d^{(1/2)}_{1/2 -1/2} = \sin^2(\beta/2) = 1/2(1-\cos\beta)$$

$$d^{(1)}_{-1+1} = d^{(1/2)}_{-1/2\, 1/2} d^{(1/2)}_{-1/2\, 1/2} = \sin^2(\beta/2) = \frac12 (1-\cos\beta)$$

$$d^{(1)}_{+10} = \sum_{\substack{m_1 m_2 \\ m_1' m_2'}} \langle 1\,1 | \tfrac12 m_1' ; \tfrac12 m_2' \rangle \langle \tfrac12 m_1 ; \tfrac12 m_2 | 1\, 0 \rangle d^{(1/2)}_{m_1' m_1} d^{(1/2)}_{m_2' m_2}$$

$$= \frac{1}{\sqrt2} d^{(1/2)}_{1/2\, 1/2} d^{(1/2)}_{1/2 -1/2} + \frac{1}{\sqrt2} d^{(1/2)}_{1/2 -1/2} d^{(1/2)}_{1/2\, 1/2}$$

$$= -\sqrt2 \cos(\beta/2)\sin(\beta/2) = -\sqrt2 \frac{\sin\beta}{2} = -\frac{1}{\sqrt2}\sin\beta$$

$$d^{(1)}_{0+1} = \frac{1}{\sqrt2} d^{(1/2)}_{-1/2\, 1/2} d^{(1/2)}_{1/2\, 1/2} + \frac{1}{\sqrt2} d^{(1/2)}_{1/2\, 1/2} d^{(1/2)}_{-1/2\, 1/2}$$

$$= \sqrt2 \sin(\beta/2)\cos(\beta/2) = \frac{1}{\sqrt2}\sin\beta$$

$$d^{(1)}_{-10} = \sum_{\substack{m_1 m_2 \\ m_1' m_2'}} \langle 1\, -1 | \tfrac12 m_1' ; \tfrac12 m_2' \rangle \langle \tfrac12 m_1 ; \tfrac12 m_2 | 1\, 0 \rangle d^{(1/2)}_{m_1' m_1} d^{(1/2)}_{m_2' m_2}$$

$$= \frac{1}{\sqrt2} d^{(1/2)}_{-1/2\, 1/2} d^{(1/2)}_{-1/2 -1/2} + \frac{1}{\sqrt2} d^{(1/2)}_{-1/2 -1/2} d^{(1/2)}_{-1/2\, 1/2}$$

$$= \sqrt2 d^{(1/2)}_{-1/2\, 1/2} d^{(1/2)}_{-1/2 -1/2} = \sqrt2 \cos(\beta/2)\sin(\beta/2) = \frac{1}{\sqrt2}\sin\beta$$

$$d^{(1)}_{0-1} = \frac{1}{\sqrt2} d^{(1/2)}_{1/2 -1/2} d^{(1/2)}_{-1/2 -1/2} + \frac{1}{\sqrt2} d^{(1/2)}_{-1/2 -1/2} d^{(1/2)}_{1/2 -1/2}$$

$$= -\frac{1}{\sqrt2}\sin\beta$$

$$d_{00}^{(1)} = \sum_{\substack{m_1 m_2 \\ m_1' m_2'}} \langle 1\ 0 | 1/2\, m_1'; 1/2\, m_2' \rangle \langle 1/2\, m_1; 1/2\, m_2 | 1\ 0 \rangle\, d^{(1/2)}_{m_1' m_1}\, d^{(1/2)}_{m_2' m_2}$$

$$= \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \left\{ d^{(1/2)}_{1/2\,1/2}\, d^{(1/2)}_{-1/2\,-1/2} + d^{(1/2)}_{1/2\,-1/2}\, d^{(1/2)}_{-1/2\,1/2} + d^{(1/2)}_{-1/2\,1/2}\, d^{(1/2)}_{1/2\,-1/2} + d^{(1/2)}_{-1/2\,-1/2}\, d^{(1/2)}_{1/2\,1/2} \right\}$$

$$= \frac{1}{2} \left\{ 2\cos^2(\beta/2) - 2\sin^2(\beta/2) \right\} = \cos^2(\beta/2) - \sin^2(\beta/2) = \cos\beta$$

Equal to the ones calculated in lecture 5.

+ 10p

20p → 100%