Disclaimer

The solution at hand was written in the course of the respective class at the University of Bonn. If not stated differently on top of the first page or the following website, the solution was prepared and handed in solely by me, Marvin Zanke. Anything in a different color than the ball pen blue is usually a correction that I or a tutor made. For more information and all my material, check:
https://www.physics-and-stuff.com/

I raise no claim to correctness and completeness of the given solutions! This equally applies to the corrections mentioned above.

14.12.2017 **Group theory Exercise 8** — Marvin Zanke

H8.1

(1) $D_4 = \{e, b, c, c^2, c^3, bc, bc^2, bc^3\}$

Have the conjugacy classes:

$[e] = \{e\}$, $[c] = \{c, c^3\}$, $[c^2] = \{c^2\}$

$[b] = \{b, bc^2\}$, $[bc] = \{bc, bc^3\}$

We know: • $N_{irr} = N_{cl} = 5$

• $N_G = 8 = |\chi^1(e)|^2 + \ldots + |\chi^5(e)|^2$

$= d_1^2 + \ldots + d_5^2 = 2^2 + 1^2 + 1^2 + 1^2 + 1^2$

$\rightsquigarrow$

	1	1	2	2	2
D_4	$[e]$	$[c^2]_{C_2}$	$[b]_{C_2}$	$[bc]_{C_2}$	$[c]_{C_4}$
1	1	1	1	1	1
$1'$	1	± 1	± 1	± 1	± 1
$1''$	1	± 1	± 1	± 1	± 1
$1'''$	1	± 1	± 1	± 1	± 1
2	2	$^{x}\pm 2$	$^{y}0$	$^{z}0$	$^{w}0$

The orthogonality theorems state that

$$(*)\quad N_G\,\delta^{\mu\nu} = \sum_{i=1}^{5} N_i\,\chi^\mu(Cl_i)\,(\chi^\nu(Cl_i))^*$$

$$(**)\quad \frac{N_G}{N_i}\,\delta_{ij} = \sum_{\nu=1}^{5} \chi^\nu(Cl_i)\,(\chi^\nu(Cl_j))^*$$

Classes that contain their own inverse: $[e]$, $[c^2]$, $[b]$, $[bc]$

$\rightsquigarrow \chi^\nu(g) = \chi^\nu(Cl_i)$ for $g \in Cl_i$

$= \chi^\nu(g^{-1}) = (\chi^\nu(g))^{-1}$ for the 1-dim rep.

χ^r can also be rational no's?

$\left(\chi^\nu(e) = \chi^\nu(g)\,\chi^\nu(g^{-1}) \Longrightarrow \chi^\nu(g) \text{ invertible}\right)$

$\rightsquigarrow \chi^\nu(g)^2 = 1 \rightsquigarrow \chi^\nu(g) = \pm 1 \;\; \forall g$ in 1 dim rep.

Then, using $(**)$, we find $8 = 4\cdot 1^2 + x^2 \rightsquigarrow x^2 = 2$ and

$4 = 4\cdot 1^2 + y^2 \rightsquigarrow y^2 = 0$ analog. for z, w

then $\chi^\nu(c^3)\overset{c.c}{=}\chi^\nu(c) = \chi^\nu(c^2)\chi^\nu(c) \rightsquigarrow \chi^\nu(c^2) = 1$ for all 1-dim.

$\Longrightarrow \chi^2_{c^2} = -2$ from $0 = \chi^1_e \chi^1_{c^2} + \dots + \chi^2_c \chi^2_{c^2}$

D_4	$\overset{1}{[e]}$	$\overset{1}{[c^2]}$	$\overset{2}{[b]}$	$\overset{2}{[bc]}$	$\overset{2}{[c]}$
1	1	1	1	1	1
1'	1	1	1	-1	-1
1''	1	1	-1	1	-1
1'''	1	1	-1	-1	1
2	2	-2	0	0	0

From $8 \overset{!}{=} 2^2 + (-2)^2 + {\chi^2_b}^2 + {\chi^2_{bc}}^2 + {\chi^2_c}^2$

$\Longrightarrow \chi_b = \chi_{bc} = \chi_c = 0$

For the remaining (± 1) entries, we notice

$0 = 1 + \chi^{1'}_b + \chi^{1''}_b + \chi^{1'''}_b$ and analogue for χ_{bc}, χ_c

$\Longrightarrow$ 2 of them (-1) and $1 \times (+1)$

As those 1-dim. irreps are exchangeable, we can use all possible combinations.

+ Gp

(8) Let $D^{(5)}$ be the 2-dim. irrep. of D_4.
$\uparrow$ 8 elements

Why index 5?

Consider $\tilde{D} = D^{(5)} \times D^{(5)}$, for which we (defined) had

$$\chi^{(5\times5)}(g) \equiv \mathrm{Tr}(\tilde{D}(g)) = \mathrm{Tr}((D^{(5)} \times D^{(5)})(g)) = \chi^{(5)}(g)\,\chi^{5}(g)$$

$$\rightsquigarrow \frac{1}{N_G} \sum_{\text{class } i} N_i |\chi^{(5\times5)}(Cl_i)|^2 = \frac{1}{8} \sum_{\text{class } i} N_i |\chi^{5}(Cl_i)|^4$$

+

Group G or G×G represented by $D^5 \times D^5$?

$$= \frac{1}{8}(1\cdot 2^4 + 1\cdot(-2)^4) = \frac{1}{8}(32) = 4 > 1$$

$\rightsquigarrow$ reducible ✓ +

Reducing $(\equiv)\times(\equiv)$ to $(\equiv)$

$$D^{(5)} \times D^{(5)} = \sum_{\oplus} q_\gamma \gamma^\gamma \quad w/\ q_\gamma = \frac{1}{N_G} \sum_{g \in D_4} \chi^\gamma(g^{-1})\,\chi^{(5)}(g)\,\chi^{(5)}(g)$$

$$\rightsquigarrow q_1 = \frac{1}{8}(1\cdot 2\cdot(+2) + 1(-2)^2) = 1$$

$$q_{1'} = \frac{1}{8}(1\cdot 2^2 + 1(-2)^2) = 1$$

$$q_{1''} = \frac{1}{8}(1\cdot 2^2 + 1(-2)^2) = 1$$

$$q_{1'''} = \frac{1}{8}(1\cdot 2^2 + 1(-2)^2) = 1$$

$$q_2 = \frac{1}{8}(2\cdot 2^2 - 2(-2)^2) = 0$$

$$\rightsquigarrow D^{(5)} \times D^{(5)} = D^1 + D^{1'} + D^{1''} + D^{1'''}$$

4P

H8.2

$Q_4 = \langle i\sigma_1, i\sigma_2 \mid \sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k \rangle$ quaternion group with 8 elements

the conjugacy classes are (see H4.2):

$[\mathbb{1}] = \{\mathbb{1}\}$, $[-\mathbb{1}] = \{-\mathbb{1}\}$, $[i\sigma_1] = \{i\sigma_1, -i\sigma_1\}$
$[i\sigma_2] = \{i\sigma_2, -i\sigma_2\}$, $[i\sigma_3] = \{i\sigma_3, -i\sigma_3\}$

- We have $N_{irr} = N_{cl} = 5$
- $N_G = 8 = |\chi^1(e)|^2 + \dots + |\chi^5(e)|^2 = d_1^2 + \dots + d_5^2$
  $= 2^2 + 1^2 + 1^2 + 1^2 + 1^2$

Q_4	$\overset{1}{[\mathbb{1}]}$	$\overset{1}{[-\mathbb{1}]}$	$\overset{2}{[i\sigma_1]}$	$\overset{2}{[i\sigma_2]}$	$\overset{2}{[i\sigma_3]}$
1	1	1	1	1	1
1'	1	1	1	-1	-1
1''	1	1	-1	1	-1
1'''	1	1	-1	-1	1
2	2	-2	0	0	0

- We again have the inverse elements in the same class as the element itself, thus:
  $\chi^\nu(g) = \chi^\nu(g^{-1}) = (\chi^\nu(g))^{-1}$ for the 1-dim. rep.
  $\Longleftrightarrow \chi^\nu(g)^2 = 1$ ($\neq 0$ as 1-dim would mean that the element is sent to 0 ...) +
- Furthermore: $\chi^\nu(-i\sigma_i) = \chi^\nu(i\sigma_i) = \chi^\nu(-\mathbb{1})\,\chi^\nu(i\sigma_i)$
  $\rightsquigarrow \chi^\nu(-\mathbb{1}) = 1$ for the 1 dim rep. +

- $0 = 4\cdot 1 + 2\cdot\chi^{(2)}_{-\mathbb{1}} \rightsquigarrow \chi^{(2)}_{-\mathbb{1}} = -2$
- $4 = 1^2 + (\pm 1)^2 + (\pm 1)^2 + (\pm 1)^2 + {\chi^2_{i\sigma_i}}^2 \rightsquigarrow \chi^2_{i\sigma_i} = 0$
- $0 = 1 + \chi^{1'}_{i\sigma_i} + \chi^{1''}_{i\sigma_i} + \chi^{1'''}_{i\sigma_i} \rightsquigarrow$ 3 solutions, $2\times(-1)$ and $1\times(+1)$
  $\rightsquigarrow$ use all, as 1', 1'', 1''' interchangeable + 7a

(2) We know that $Q_4 \not\cong D_4$, but they obviously have the same character table. From this, we can easily derive that the character table does <u>NOT</u> specify a group uniquely!

+ 3p

20p → 100% Wow M.K.